

# REDUCING CUBICAL SET MODELS OF CONCURRENT SYSTEMS

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**ABSTRACT.** Precubical sets, i.e., cubical sets without degeneracies, can be used to model systems of concurrently executing processes. From the point of view of concurrency theory, two precubical sets can be considered equivalent if their geometric realizations have the same directed homotopy type relative to the extremal elements in the sense of P. Bubenik. We give conditions under which it is possible to collapse an edge or to eliminate a cube and a free face in a 2-dimensional precubical set to obtain an equivalent smaller one. We use our results to construct small models of some simple precubical sets.

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## 1. INTRODUCTION

It is known for some time now that precubical sets, i.e., cubical sets without degeneracies, can be used to model concurrent systems (cf. [2], [3], [5], [6], [7]). These are systems of two or more computational processes which may communicate, share resources, and execute in parallel. Let us consider, as an example, a very simple concurrent system where two processes  $A$  and  $B$  write to a piece of shared memory. Each process performs a sequence of three actions: it accesses the memory, writes its name, and terminates. The processes execute simultaneously but cannot write to the memory at the same time. This situation can be modeled by the 2-dimensional precubical set depicted in figure 1(a) (the definition of precubical sets is recalled in 2.1). The vertices represent the states of the system, the horizontal arrows represent the actions of process  $A$ , and the vertical arrows represent the actions of process  $B$ . Moreover, if it does not matter in which order an action of  $A$  and an action of  $B$  are executed and they may actually be performed concurrently, then this is indicated by a square linking the two pairs of arrows corresponding to a consecutive execution of the actions. The precubical set has a hole reflecting the fact that only one process can write its name to the memory at a time.

Any precubical set can be realized geometrically as a topological space and indeed even as a d-space in the sense of M. Grandis [9]. A d-space is a topological space with a distinguished set of paths, called d-paths, which equip the space with a direction of time. The d-paths in the geometric realization of a precubical set are obtained by pasting together increasing paths on cubes. In the interpretation of a precubical set as a model of a concurrent system, the passage to the geometric realization adds all possible intermediate states of the system to the model. The d-paths represent complete or partial executions of the system. We remark that with d-paths it is possible to model not only consecutive but also truly concurrent executions of independent actions.

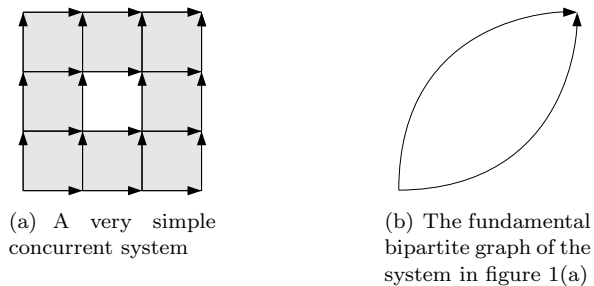


FIGURE 1.

Consider again our example concurrent system. There exists an infinite number of d-paths leading from the initial state in the lower left corner to the final state in the upper right corner. Computer scientifically, two such d-paths can be considered equivalent if they represent executions which produce the same result, i.e., executions where the processes write to the memory in the same order. Geometrically, this happens precisely when the d-paths turn around the hole on the same side. This leads to the following notion of equivalence of d-paths: Two d-paths  $\alpha$  and  $\beta$  from a point  $x$  in a d-space to a point  $y$  are said to be dihomotopic relative to  $\{0, 1\}$  if there exists a homotopy  $H$  from  $\alpha$  to  $\beta$  such that each path  $H(-, t)$  is a d-path from  $x$  to  $y$ . In the example, there are two relative dihomotopy classes of d-paths leading from the initial to the final state of the system corresponding to the two possible orders in which the processes can write to the memory.

An important tool in the study of the directed structure of a d-space is its fundamental category (cf. [6], [9]). This is the directed analogue of the fundamental groupoid of a topological space. The objects of the fundamental category of a d-space are its points and the morphisms are the relative dihomotopy classes of d-paths. The fundamental category of a d-space is, of course, a huge object and a main line of research in the area of directed algebraic topology is the development of methods to extract the essential information of the fundamental category (cf. [1], [3], [4], [8], [9, I.3], [10]). A basic construction in this context is P. Bubenik's fundamental bipartite graph of a d-space which is the full subcategory of the fundamental category generated by the so-called extremal elements (cf. [1]). In the geometric realization of a precubical set, the extremal elements are the points which correspond to the vertices in which no edge begins or no edge ends (cf. 3.5). The fundamental bipartite graph of a d-space model of a concurrent system represents the essential execution schedules of the system. In the case of our example system, the fundamental bipartite graph is indicated in figure 1(b).

The purpose of this paper is to establish some results which permit us to reduce a given precubical set to an equivalent smaller one. This approach complements the strategy to replace the fundamental category by a smaller object containing the relevant directed information. In this paper, we consider two precubical sets as equivalent if their geometric realizations are dihomotopy equivalent relative to the extremal elements. The notion of dihomotopy equivalence we use here is based on a straightforward extension of the notion of dihomotopy of d-paths to d-maps, i.e., morphisms of d-spaces (cf. 3.1). The reader should note, however, that other concepts of directed homotopy equivalence have been defined (cf. [5], [10]) and that

it is not clear at the time being what the ultimate notion of equivalence for topological models of concurrent systems is. We point out that all reductions based on our results are guaranteed to preserve the fundamental bipartite graph. Being dihomotopy equivalent relative to the extremal elements is, however, a much stronger condition than having the same fundamental bipartite graph. Using our techniques, the precubical set represented in figure 1(a) can be reduced to a precubical set that looks exactly like the fundamental bipartite graph in figure 1(b).

If a precubical set admits a cube with a free face, then this cube and the free face can be eliminated without changing the ordinary homotopy type of the geometric realization of the precubical set. Unfortunately, the directed homotopy type will change in general under such a collapsing operation. In this paper, we give conditions under which it is possible to locally reduce or simplify precubical sets without changing the dihomotopy type relative to the extremal elements of the geometric realization. We restrict ourselves to the 2-dimensional case and leave the more complicated case of higher dimensions for future work. From the point of view of applications the 2-dimensional case is particularly important. Indeed, many classical models for concurrency can be interpreted as asynchronous transition systems (cf. [11]) and these, in turn, can be transformed into 2-dimensional precubical sets (cf. [7]).

The main results are contained in sections 5 and 6. In section 2, we collect some basic facts on precubical sets and d-spaces. Section 3 is devoted to dihomotopy and the fundamental bipartite graph. In section 4, we present a method to construct d-maps and dihomotopies on geometric realizations of precubical sets. The last section contains some small examples.

## 2. PRECUBICAL SETS AND d-SPACES

**Definition 2.1.** A *precubical set* is a graded set  $P = (P_n)_{n \geq 0}$  with *boundary operators*  $d_i^k : P_n \rightarrow P_{n-1}$  ( $n > 0$ ,  $k = 0, 1$ ,  $i = 1, \dots, n$ ) satisfying the relations  $d_i^k \circ d_j^l = d_{j-1}^l \circ d_i^k$  ( $k, l = 0, 1$ ,  $i < j$ ). If there exists a largest  $n$  such that  $P_n \neq \emptyset$ , then this  $n$  is called the *dimension* of  $P$ . The degree of an element  $x$  of  $P$  will be denoted by  $|x|$ . The elements of degree 0 are also called the *vertices* of  $P$ . A morphism of precubical sets is a morphism of graded sets which is compatible with the boundary operators. The category of precubical sets will be denoted by  $\square\mathbf{Set}$ .

The category  $\square\mathbf{Set}$  can be seen as the presheaf category of functors  $\square^{op} \rightarrow \mathbf{Set}$  where  $\square$  is the small subcategory of  $\mathbf{Top}$  whose objects are the standard  $n$ -cubes  $I^n$  ( $n \geq 0$ ) and whose non-identity morphisms are composites of the maps  $\delta_i^k : I^n \rightarrow I^{n+1}$  ( $n \geq 0$ ,  $i \in \{1, \dots, n+1\}$ ,  $k \in \{0, 1\}$ ) given by  $\delta_i^k(u_1, \dots, u_n) = (u_1, \dots, u_{i-1}, k, u_i, \dots, u_n)$ . Here, we consider the 0-cube as the one-point space  $I^0 = \{()\}$ . The *precubical  $n$ -cube* is the  $n$ -dimensional precubical set  $\mathbb{I}^n = \square(-, I^n)$ . By Yoneda's Lemma, an element  $x$  of degree  $n$  of a precubical set  $P$  determines a unique morphism of precubical sets  $x_\# : \mathbb{I}^n \rightarrow P$  such that  $x_\#(id_{I^n}) = x$ .

**Definition 2.2.** Let  $P$  be a precubical set and  $x \in P_n$  be an element. We say that  $x$  is *regular* if the morphism  $x_\#$  is injective.

We remark that if  $x$  is regular, then so is each  $d_i^k x$ .

**Definition 2.3.** A *precubical subset* of a precubical set  $P$  is a precubical set  $Q$  such that  $Q_n \subseteq P_n$  for all  $n \geq 0$  and the boundary operators of  $Q$  and  $P$  coincide

on  $Q$ . The *opposite precubical set* of a precubical set  $P$  with boundary operators  $d_i^k$  is the precubical set  $P^{op}$  with boundary operators  $\partial_i^k$  defined by  $P_r^{op} = P_r$  and  $\partial_i^k = d_i^{1-k}$ . The *transposed precubical set* of a precubical set  $P$  with boundary operators  $d_i^k$  is the precubical set  $P^t$  with boundary operators  $\partial_i^k$  defined by  $P_r^t = P_r$  and  $\partial_i^k = d_{r+1-i}^k: P_r \rightarrow P_{r-1}$  ( $i \in \{1, \dots, r\}$ ). The maps  $P \mapsto P^{op}$  and  $P \mapsto P^t$  extend to functors in the obvious way.

The terminology of opposite and transposed precubical sets is an adaptation of the one used in [9] for cubical sets. Note that the opposite and transposed precubical set functors are involutions and that they preserve precubical subsets. Note also that a regular element of a precubical set  $P$  is also regular as an element of  $P^{op}$  and  $P^t$ . Note finally that the image of a morphism  $f: P \rightarrow Q$  of precubical sets is a precubical subset of  $Q$ . If  $f$  is injective, then  $f$  restricts to an isomorphism  $P \xrightarrow{\cong} f(P)$ .

Precubical sets can be realized geometrically as d-spaces in the sense of M. Grandis [9]. These spaces are defined as follows:

**Definition 2.4.** [9, I.1.4] A *d-space* is a topological space  $X$  together with a subset  $dX \subseteq X^I$  such that

- (i)  $dX$  contains all constant paths,
- (ii)  $dX$  is closed under composition with (not necessarily strictly) increasing maps  $I \rightarrow I$ ,
- (iii)  $dX$  is closed under concatenation.

The elements of  $dX$  are called *d-paths* in  $X$ . A *d-map* is a continuous map  $f: X \rightarrow Y$  between d-spaces such that for any  $\omega \in dX$ ,  $f \circ \omega \in dY$ . The category of d-spaces and d-maps is denoted by **dTop**.

**Example 2.5.** The *directed interval* is the d-space  $\vec{I} = (I, d\vec{I})$  where  $d\vec{I}$  consists of the (not necessarily strictly) increasing maps  $I \rightarrow I$ . The d-paths in a d-space  $X$  are precisely the d-maps  $\vec{I} \rightarrow X$ . Taking the  $n$ -fold product of  $\vec{I}$  with itself we obtain the *directed  $n$ -cube*  $\vec{I}^n$ . Note that the product of two d-spaces  $X$  and  $Y$  is a d-space with respect to the set  $d(X \times Y)$  corresponding to  $dX \times dY$  under the bijection  $(X \times Y)^I \approx X^I \times Y^I$ .

**Remark 2.6.** We occasionally need the following criterion for d-paths: A path  $\omega$  in a d-space  $X$  is a d-path if there exist  $0 = b_0 < b_1 < \dots < b_l = 1$  such that for each  $i \in \{1, \dots, l\}$  the path  $I \rightarrow X$ ,  $t \mapsto \omega((1-t)b_{i-1} + tb_i)$  is a d-path. Indeed, the concatenation of these d-paths is the composite  $\omega \circ \phi$  where  $\phi: I \rightarrow I$  is the increasing homeomorphism given by  $\phi(t) = (i-lt)b_{i-1} + (lt-i+1)b_i$ ,  $1 \leq i \leq l$ ,  $\frac{i-1}{l} \leq t \leq \frac{i}{l}$ . Since  $\omega$  is an increasing reparametrization of  $\omega \circ \phi$ , it is a d-path.

**Definition 2.7.** (cp. [3], [5], [6], [9, I.1.6.7]) The *geometric realization* of a precubical set  $P$  is the quotient space  $|P| = (\coprod_{n \geq 0} P_n \times I^n) / \sim$  where the sets  $P_n$  are considered as discrete spaces and the equivalence relation is given by

$$(d_i^k x, u) \sim (x, \delta_i^k(u)), \quad x \in P_{n+1}, \quad u \in I^n, \quad i \in \{1, \dots, n+1\}, \quad k \in \{0, 1\}.$$

The geometric realization  $|P|$  is a d-space with respect to the set  $d|P|$  consisting of increasing reparametrizations of finite concatenations of paths  $\omega: I \rightarrow |P|$  of the form  $\omega(t) = [x, \alpha(t)]$  where  $x \in P_n$  and  $\alpha$  is a continuous map  $I \rightarrow I^n$  which is order-preserving with respect to the natural order of  $I$  and the componentwise

natural order of  $I^n$ . The geometric realization of a morphism of precubical sets  $f: P \rightarrow Q$  is the d-map  $|f|: |P| \rightarrow |Q|$  given by  $|f|([x, u]) = [f(x), u]$ . With these definitions the geometric realization is a functor  $|\cdot|: \square\mathbf{Set} \rightarrow \mathbf{dTop}$ .

**Example 2.8.** The map  $\vec{I}^n \rightarrow |\mathbb{I}^n|$ ,  $u \mapsto [id_{I^n}, u]$  is an isomorphism of d-spaces.

We remark that the geometric realization of a precubical set  $P$  is a CW-complex (cf. [5]). The  $n$ -skeleton of  $|P|$  is the geometric realization of the  $n$ -dimensional precubical subset  $P_{\leq n}$  of  $P$  defined by  $(P_{\leq n})_m = P_m$  ( $m \leq n$ ). The closed  $n$ -cells of  $|P|$  are the d-spaces  $|x_{\sharp}(\mathbb{I}^n)|$  where  $x \in P_n$ . The characteristic map of the cell  $|x_{\sharp}(\mathbb{I}^n)|$  is the d-map  $\vec{I}^n \xrightarrow{\cong} |\mathbb{I}^n| \xrightarrow{|x_{\sharp}|} |P|$  and this map is an isomorphism onto its image if and only if  $x$  is regular. Note also that the geometric realization of a precubical subset  $Q$  of  $P$  is a CW-subcomplex of  $|P|$ . We shall show that it is also a d-subspace of  $|P|$  in the sense of the following definition:

**Definition 2.9.** [9, I.1.4.1] A *d-subspace* of a d-space  $X$  is a d-space  $A$  such that the topological space  $A$  is a subspace of  $X$  and  $dA = \{\omega \in A^I \mid (A \hookrightarrow X) \circ \omega \in dX\}$ .

**Remark 2.10.** Note that if  $A$  is a d-subspace of  $X$ , then the inclusion  $A \hookrightarrow X$  is a d-map. Note also that if  $f: X \rightarrow Y$  is a d-map and  $A \subseteq X$  and  $B \subseteq Y$  are d-subspaces such that  $f(A) \subseteq B$ , then  $f$  restricts to a d-map  $A \rightarrow B$ .

**Proposition 2.11.** *Let  $Q$  be a precubical subset of a precubical set  $P$ . Then  $|Q|$  is a d-subspace of  $|P|$ .*

*Proof.* Since  $|Q|$  is a CW-subcomplex of  $|P|$ , we only have to show that  $d|Q| = \{\omega \in |Q|^I \mid \iota \circ \omega \in d|P|\}$  where  $\iota$  is the inclusion  $|Q| \hookrightarrow |P|$ . Since  $\iota$  is a d-map,  $d|Q| \subseteq \{\omega \in |Q|^I \mid \iota \circ \omega \in d|P|\}$ . Consider an element  $x \in P_n$  and a path  $\nu \in d|P|$  of the form  $\nu(t) = [x, \alpha(t)]$  where  $\alpha: I \rightarrow I^n$  is order-preserving and suppose that  $\nu(I) \subseteq |Q|$ . The result follows if we can show that  $\nu$ , seen as a path in  $|Q|$ , belongs to  $d|Q|$ . We proceed by induction on  $n$ . If  $n = 0$ , then  $\nu$  is a constant path and therefore as a path in  $|Q|$ ,  $\nu \in d|Q|$ . Suppose that  $n \geq 1$  and that the claim holds for  $n - 1$ . If  $x \in Q_n$ , then there is nothing to show. Suppose that  $x \notin Q_n$ . Then  $\alpha(I) \subseteq I^n \setminus ]0, 1[^n$ . Consider the set

$$S = \{s \in I \mid \alpha(s) \in \bigcup_{i=1}^n \delta_i^0(I^{n-1})\}.$$

Suppose first that  $S = \emptyset$ . Then there exists an  $i \in \{1, \dots, n\}$  such that  $\alpha(0)_i = 1$ . Since  $\alpha$  is order-preserving, this implies that  $\alpha(I) \subseteq \delta_i^1(I^{n-1})$  and hence that  $\alpha = \delta_i^1 \circ \beta$  for the order-preserving map  $\beta: I \rightarrow I^{n-1}$  given by  $\beta(t) = (\alpha(t)_1, \dots, \alpha(t)_{i-1}, \alpha(t)_{i+1}, \dots, \alpha(t)_n)$ . It follows that  $\nu(t) = [d_i^1 x, \beta(t)]$  ( $t \in I$ ) and hence, by the inductive hypothesis, that as a path in  $|Q|$ ,  $\nu \in d|Q|$ . Suppose next that  $S \neq \emptyset$  and that  $\sup S = 1$ . Since  $\alpha$  is continuous,  $1 \in S$  and  $\alpha(1)_i = 0$  for some  $i \in \{1, \dots, n\}$ . Since  $\alpha$  is order-preserving, this implies that  $\alpha(I) \subseteq \delta_i^0(I^{n-1})$  and hence that  $\alpha = \delta_i^0 \circ \beta$  for the order-preserving map  $\beta: I \rightarrow I^{n-1}$  given by  $\beta(t) = (\alpha(t)_1, \dots, \alpha(t)_{i-1}, \alpha(t)_{i+1}, \dots, \alpha(t)_n)$ . It follows that  $\nu(t) = [d_i^0 x, \beta(t)]$  ( $t \in I$ ) and hence, by the inductive hypothesis, that as a path in  $|Q|$ ,  $\nu \in d|Q|$ . Suppose finally that  $S \neq \emptyset$  and that  $\sup S < 1$ . Set  $b = \sup S$ . Since  $\alpha$  is continuous,  $b \in S$  and  $\alpha(b)_i = 0$  for some  $i \in \{1, \dots, n\}$ . Since  $\alpha$  is order-preserving, this implies that  $\alpha([0, b]) \subseteq \delta_i^0(I^{n-1})$ . We have  $\alpha([b, 1]) \subseteq \bigcup_{j=1}^n \delta_j^1(I^{n-1})$ . Since

$\alpha$  is continuous and  $b < 1$ , this implies that  $\alpha([b, 1]) \subseteq \bigcup_{j=1}^n \delta_j^1(I^{n-1})$ . Therefore  $\alpha(b)_j = 1$  for some  $j \in \{1, \dots, n\}$ . Since  $\alpha$  is order-preserving, it follows that  $\alpha([b, 1]) \subseteq \delta_j^1(I^{n-1})$ . If  $b = 0$ , the same argument as in the case  $S = \emptyset$  shows that as a path in  $|Q|$ ,  $\nu \in d|Q|$ . If  $b > 0$ , consider the paths  $\nu_1, \nu_2: I \rightarrow |P|$  defined by  $\nu_1(t) = \nu(tb)$  and  $\nu_2(t) = \nu((1-t)b + t)$ . Then for all  $t \in I$ ,  $\nu_1(t) = [d_i^0 x, \beta(t)]$  where  $\beta$  is the order-preserving map defined by

$$\beta(t) = (\alpha(tb)_1, \dots, \alpha(tb)_{i-1}, \alpha(tb)_{i+1}, \dots, \alpha(tb)_n)$$

and  $\nu_2(t) = [d_j^1 x, \gamma(t)]$  where  $\gamma$  is the order-preserving map defined by

$$\gamma(t) = (\alpha((1-t)b + t)_1, \dots, \alpha((1-t)b + t)_{j-1}, \alpha((1-t)b + t)_{j+1}, \dots, \alpha((1-t)b + t)_n).$$

By the inductive hypothesis, as paths in  $|Q|$ ,  $\nu_1, \nu_2 \in d|Q|$ . It follows then from 2.6 that as a path in  $|Q|$ ,  $\nu \in d|Q|$ .  $\square$

Reversing the direction of the d-paths of a d-space one obtains the opposite d-space:

**Definition 2.12.** [9, I.1.4.0] Given a path  $\omega: I \rightarrow X$  we denote the inverse path  $I \rightarrow X$ ,  $t \mapsto \omega(1-t)$  by  $\bar{\omega}$ . The *opposite d-space* of a d-space  $X = (X, dX)$  is the d-space  $X^{op} = (X, dX^{op})$  defined by  $dX^{op} = \{\bar{\omega} \mid \omega \in dX\}$ . The map  $X \mapsto X^{op}$  extends to a functor in the obvious way.

Note that the opposite d-space functor is an involution. Note also that if  $A$  is a d-subspace of  $X$ , then  $A^{op}$  is a d-subspace of  $X^{op}$ .

**Proposition 2.13.** *Let  $P$  be a precubical set. A natural isomorphism of d-spaces  $\phi_P: |P|^{op} \rightarrow |P^{op}|$  is given by  $[z, (u_1, \dots, u_r)] \mapsto [z, (1-u_1, \dots, 1-u_r)]$ .*

*Proof.* The continuous map  $\phi_P$  is a natural homeomorphism with inverse  $\phi_P^{-1} = (\phi_{P^{op}})^{op}$ . In order to check that  $\phi_P$  is a morphism of d-spaces let first  $\omega \in d|P|$  be a path of the form  $\omega(t) = [z, \alpha(t)]$  where  $z \in P_r$  and  $\alpha: I \rightarrow I^r$  is order-preserving. Consider the order-preserving path  $\beta: I \rightarrow I^r$  defined by

$$\beta(t) = (1 - \bar{\alpha}_1(t), \dots, 1 - \bar{\alpha}_r(t)).$$

We have

$$\phi_P \circ \bar{\omega}(t) = \phi_P(\omega(1-t)) = \phi_P([z, \alpha(1-t)]) = \phi_P([z, \bar{\alpha}(t)]) = [z, \beta(t)]$$

and hence  $\phi_P \circ \bar{\omega} \in d|P^{op}|$ . Now consider an increasing map  $\gamma: I \rightarrow I$  and the concatenation  $\omega_1 * \dots * \omega_l$  of  $l$  paths  $\omega_1, \dots, \omega_l \in d|P|$  such that  $\phi_P \circ \bar{\omega}_i \in d|P^{op}|$  for each  $i \in \{1, \dots, l\}$ . One easily checks that

$$\phi_P \circ \overline{(\omega_1 * \dots * \omega_l)} \circ \gamma = ((\phi_P \circ \bar{\omega}_1) * \dots * (\phi_P \circ \bar{\omega}_l)) \circ (1 - \bar{\gamma}).$$

It follows that  $\phi_P \circ \overline{(\omega_1 * \dots * \omega_l)} \circ \gamma \in d|P^{op}|$ . Thus,  $\phi_P \circ \bar{\omega} \in d|P^{op}|$  for any  $\omega \in d|P|$  and  $\phi_P$  is a d-map.  $\square$

For transposed precubical sets we have the following proposition:

**Proposition 2.14.** *Let  $P$  be a precubical set. A natural isomorphism of d-spaces  $\sigma_P: |P| \rightarrow |P^t|$  is given by  $[z, (u_1, \dots, u_r)] \mapsto [z, (u_r, \dots, u_1)]$ .*

*Proof.* One easily checks that  $\sigma_P$  is a well-defined natural homeomorphism with inverse  $\sigma_P^{-1} = \sigma_{P^t}$ . Let  $\omega \in d|P|$  be a path of the form  $\omega(t) = [z, \alpha(t)]$  where  $z \in P_r$  and  $\alpha: I \rightarrow I^r$  is order-preserving. Consider the order-preserving path  $\beta: I \rightarrow I^r$  given by  $\beta(t) = (\alpha_r(t), \dots, \alpha_1(t))$ . Then  $\sigma_P \circ \omega: I \rightarrow |P^t|$  is given by  $\sigma_P \circ \omega(t) = [z, \beta(t)]$ . Thus,  $\sigma_P \circ \omega \in d|P^t|$ . Since  $d|P^t|$  is closed under finite concatenation and increasing reparametrization, it follows that  $\sigma_P$  is a d-map and hence that it is an isomorphism of d-spaces.  $\square$

### 3. DIHOMOTOPY AND THE FUNDAMENTAL BIPARTITE GRAPH

We shall work with the following notion of directed homotopy:

**Definition 3.1.** Two d-maps  $f, g: X \rightarrow Y$  are said to be *dihomotopic* if there exists a homotopy  $H: X \times I \rightarrow Y$  from  $f$  to  $g$  such that each map  $H(-, t)$  is a d-map. Such a homotopy is called a *dihomotopy* from  $f$  to  $g$ . If  $f$  and  $g$  coincide on a d-subspace  $A \subseteq X$ , then  $f$  and  $g$  are said to be *dihomotopic relative to  $A$*  if there exists a *dihomotopy relative to  $A$*  from  $f$  to  $g$ , i.e., a dihomotopy  $H: X \times I \rightarrow Y$  from  $f$  to  $g$  such that each map  $H(-, t)$  coincides with  $f$  and  $g$  on  $A$ . Let  $X$  and  $Y$  be d-spaces with a common d-subspace  $A$ . A d-map  $f: X \rightarrow Y$  satisfying  $f(a) = a$  for all  $a \in A$  is said to be a *dihomotopy equivalence relative to  $A$*  if there exists a *dihomotopy inverse relative to  $A$*  of  $f$ , i.e., a d-map  $g: Y \rightarrow X$  such that  $g(a) = a$  for all  $a \in A$  and such that  $g \circ f$  and  $f \circ g$  are dihomotopic relative to  $A$  to the identities of  $X$  and  $Y$ , respectively. A *dihomotopy equivalence* is a d-map which is dihomotopy equivalence relative to the empty d-space. Two d-spaces  $X$  and  $Y$  with a common d-subspace  $A$  are said to be *dihomotopy equivalent relative to  $A$*  if there exists a dihomotopy equivalence relative to  $A$  between them. Two d-spaces  $X$  and  $Y$  are *dihomotopy equivalent* if they are dihomotopy equivalent relative to the empty d-space.

We remark that (relative) dihomotopy is an equivalence relation which is compatible with the composition of d-maps. Some authors, as for instance M. Grandis [9], work with a stronger notion of directed homotopy, called *d-homotopy*, where the homotopies are required to be d-maps  $X \times \vec{I} \rightarrow Y$ . The reader is referred to L. Fajstrup [2] for a result concerning the equivalence of the two notions of directed homotopy for directed paths.

The one-dimensional information of a d-space is contained in its fundamental category which is the directed analogue of the fundamental groupoid of a topological space.

**Definition 3.2.** ([6], [9, I.3]) The *fundamental category* of a d-space  $X$  is the category  $\pi_1(X)$  defined as follows: The objects are the elements of  $X$  and the set of morphisms from an element  $x$  to an element  $y$  is the set of dihomotopy classes relative to  $\{0, 1\}$  of d-paths from  $x$  to  $y$ . The map  $X \mapsto \pi_1(X)$  extends in the obvious way to a functor from **dTop** to the category of small categories.

The fundamental category of a d-space is of course a huge object. This led to the development of several methods to extract the essential directed information of the fundamental category (cf. [1], [3], [4], [8], [9, I.3], [10]). In [1], P. Bubenik introduced the fundamental bipartite graph of a d-space:

**Definition 3.3.** [1] An element  $a$  of a d-space  $X$  is said to be *minimal* (*maximal*) if any morphism in  $\pi_1(X)$  with target (source)  $a$  has source (target)  $a$ . An element of  $X$  is *extremal* if it is minimal or maximal. The d-subspace of  $X$  consisting of the extremal elements is denoted by  $Extrl(X)$ . The *fundamental bipartite graph* of a d-space  $X$ , denoted by  $\pi_1(X, Extrl(X))$ , is the full subcategory of  $\pi_1(X)$  generated by  $Extrl(X)$ .

Note that the fundamental bipartite graph of a d-space is a bipartite graph if one ignores the identity morphisms. In a d-space model of a concurrent system, initial and unreachable states of the system are modeled by minimal elements and final states and deadlocks are modeled by maximal elements. The fundamental bipartite graph of the d-space represents the essential execution schedules between these critical states of the system. For precubical sets there is another definition of minimal, maximal, and extremal elements:

**Definition 3.4.** Let  $P$  be a precubical set. An element  $v \in P_0$  is said to be *minimal* (*maximal*) if there is no element  $x \in P_1$  such that  $d_1^1 x = v$  ( $d_1^0 x = v$ ). An element of  $P$  is *extremal* if it is minimal or maximal. The 0-dimensional precubical subset of  $P$  consisting of the extremal elements is denoted by  $Extrl(P)$ .

We remark that  $Extrl(P^{op}) = Extrl(P^t) = Extrl(P)$ .

**Proposition 3.5.** Let  $P$  be a precubical set. An element  $a \in |P|$  is *minimal* (*maximal*) if and only if there exists a *minimal* (*maximal*) element  $v \in P_0$  such that  $a = [v, ()]$ . Consequently,  $|Extrl(P)| = Extrl(|P|)$ .

*Proof.* Let first  $a \in |P|$  be a minimal element. Then there exists  $v \in P_0$  such that  $a = [v, ()]$ . Indeed, otherwise one would have  $a = [z, (u_1, \dots, u_r)]$  for some  $r > 0$ ,  $z \in P_r$ , and  $(u_1, \dots, u_r) \in ]0, 1[^r$ . But then the class of the d-path  $\omega: \vec{I} \rightarrow |P|$ ,  $\omega(t) = [z, t(u_1, \dots, u_r)]$  would be a morphism in  $\pi_1(X)$  with target  $a$  and source  $[z, (0, \dots, 0)] \neq a$ . Since  $a$  is minimal, such a morphism does not exist and  $a = [v, ()]$  for some  $v \in P_0$ . Suppose that  $v$  is not minimal. Then there exists an element  $x \in P_1$  such that  $d_1^1 x = v$ . Consider the d-path  $\omega: \vec{I} \rightarrow |P|$  defined by  $\omega(t) = [x, \frac{1+t}{2}]$ . Then in  $\pi_1(X)$ ,  $[\omega]: [x, \frac{1}{2}] \rightarrow a$  and hence  $[x, \frac{1}{2}] = [v, ()]$ . This is impossible and  $v$  is minimal.

Suppose now that  $a = [v, ()]$  for some minimal element  $v \in P_0$ . Let  $\omega: \vec{I} \rightarrow |P|$  be a d-path such that  $\omega(1) = a$ . We have to show that  $\omega(0) = a$ . Since  $\omega \in d|P|$ , it is an increasing reparametrization of a finite concatenation of paths of the form  $t \mapsto [z, \alpha(t)]$  where  $z \in P_r$  and  $\alpha: I \rightarrow I^r$  is order-preserving. We may therefore suppose that  $\omega$  is of this form. Then  $\omega(t) = [z, \alpha(t)]$  and  $\omega(1) = [z, \alpha(1)] = a = [v, ()]$ . If  $r = 0$ , then  $z = v$  and  $\omega(0) = [z, \alpha(0)] = [z, ()] = [v, ()] = a$ . Suppose now that  $r > 0$ . Then  $\alpha(1)_i \in \{0, 1\}$  for all  $i \in \{1, \dots, r\}$  and  $v = d_1^{\alpha(1)_r} \dots d_1^{\alpha(1)_1} z$ . Therefore

$$v = d_1^{\alpha(1)_i} d_2^{\alpha(1)_r} \dots d_2^{\alpha(1)_{i+1}} d_1^{\alpha(1)_{i-1}} \dots d_1^{\alpha(1)_1} z$$

for all  $i \in \{1, \dots, r\}$ . Since  $v$  is minimal, we obtain that  $\alpha(1)_i = 0$  for all  $i \in \{1, \dots, r\}$  and hence that  $\alpha(t) = (0, \dots, 0)$  for all  $t \in I$ . It follows that  $\omega(0) = [z, \alpha(0)] = [z, \alpha(1)] = \omega(1) = a$ .

The proof of the statement for maximal elements is analogous.  $\square$

We have the following result on the dihomotopy invariance of the fundamental bipartite graph:



**Theorem 3.6.** *Let  $X$  and  $Y$  be two d-spaces such that  $\text{Extrl}(X) = \text{Extrl}(Y)$  and let  $f: X \rightarrow Y$  be a dihomotopy equivalence relative to  $\text{Extrl}(X)$ . Then the functor  $\bar{\pi}_1(f): \bar{\pi}_1(X) \rightarrow \bar{\pi}_1(Y)$  restricts to an isomorphism of fundamental bipartite graphs  $\bar{\pi}_1(X, \text{Extrl}(X)) \rightarrow \bar{\pi}_1(Y, \text{Extrl}(Y))$ .*

*Proof.* Let  $g: Y \rightarrow X$  be a dihomotopy inverse relative to  $\text{Extrl}(X) = \text{Extrl}(Y)$  of  $f$ . The functors  $\bar{\pi}_1(f): \bar{\pi}_1(X) \rightarrow \bar{\pi}_1(Y)$  and  $\bar{\pi}_1(g): \bar{\pi}_1(Y) \rightarrow \bar{\pi}_1(X)$  restrict to functors  $f_*: \bar{\pi}_1(X, \text{Extrl}(X)) \rightarrow \bar{\pi}_1(Y, \text{Extrl}(Y))$  and  $g_*: \bar{\pi}_1(Y, \text{Extrl}(Y)) \rightarrow \bar{\pi}_1(X, \text{Extrl}(X))$  which are the identity on objects. Let  $a$  and  $b$  be extremal elements of  $X$  and  $\omega: \vec{I} \rightarrow X$  be a d-path from  $a$  to  $b$ . Since  $g \circ f$  is dihomotopic relative to  $\text{Extrl}(X)$  to  $\text{id}_X$ ,  $g \circ f \circ \omega$  is dihomotopic relative to  $\{0, 1\}$  to  $\omega$ . Hence  $g_* \circ f_* = \text{id}_{\bar{\pi}_1(X, \text{Extrl}(X))}$ . Similarly,  $f_* \circ g_* = \text{id}_{\bar{\pi}_1(Y, \text{Extrl}(Y))}$ .  $\square$

**Corollary 3.7.** *Let  $P$  and  $Q$  be two precubical sets such that  $\text{Extrl}(P) = \text{Extrl}(Q)$  and let  $f: |P| \rightarrow |Q|$  be a dihomotopy equivalence relative to  $|\text{Extrl}(P)|$ . Then the functor  $\bar{\pi}_1(f): \bar{\pi}_1(|P|) \rightarrow \bar{\pi}_1(|Q|)$  restricts to an isomorphism of fundamental bipartite graphs  $\bar{\pi}_1(|P|, \text{Extrl}(|P|)) \rightarrow \bar{\pi}_1(|Q|, \text{Extrl}(|Q|))$ .*

The last proposition of this section permits us to dualize results on dihomotopy equivalences for precubical sets.

**Proposition 3.8.** *Let  $P$  and  $Q$  be two precubical sets with a common precubical subset  $R$ .*

- (i) *If  $f: |P^{op}| \rightarrow |Q^{op}|$  is a dihomotopy equivalence relative to  $|R^{op}|$ , then  $(\phi_Q^{-1} \circ f \circ \phi_P)^{op}: |P| \rightarrow |Q|$  is a dihomotopy equivalence relative to  $|R|$ .*
- (ii) *If  $f: |P^t| \rightarrow |Q^t|$  is a dihomotopy equivalence relative to  $|R^t|$ , then  $\sigma_Q^{-1} \circ f \circ \sigma_P: |P| \rightarrow |Q|$  is a dihomotopy equivalence relative to  $|R|$ .*

*Proof.* (i) Note that  $(\phi_Q^{-1} \circ f \circ \phi_P)^{op}|_{|R|}$  is the inclusion  $|R| \hookrightarrow |Q|$ . Let  $g: |Q^{op}| \rightarrow |P^{op}|$  be a dihomotopy inverse relative to  $|R^{op}|$  of  $f$ ,  $H$  be a dihomotopy relative to  $|R^{op}|$  from  $\text{id}_{|P^{op}|}$  to  $g \circ f$ , and  $G$  be a dihomotopy relative to  $|R^{op}|$  from  $\text{id}_{|Q^{op}|}$  to  $f \circ g$ . Then the d-map  $(\phi_P^{-1} \circ g \circ \phi_Q)^{op}: |Q| \rightarrow |P|$  is a dihomotopy inverse relative to  $|R|$  of  $(\phi_Q^{-1} \circ f \circ \phi_P)^{op}$ . Indeed,  $(\phi_P^{-1} \circ g \circ \phi_Q)^{op}|_{|R|}$  is the inclusion  $|R| \hookrightarrow |P|$ , the continuous map  $\phi_P^{-1} \circ H \circ (\phi_P \times \text{id}_I): |P| \times I \rightarrow |P|$  is a dihomotopy relative to  $|R|$  from  $\text{id}_{|P|}$  to  $(\phi_P^{-1} \circ g \circ \phi_Q)^{op} \circ (\phi_Q^{-1} \circ f \circ \phi_P)^{op}$ , and the continuous map  $\phi_Q^{-1} \circ G \circ (\phi_Q \times \text{id}_I): |Q| \times I \rightarrow |Q|$  is a dihomotopy relative to  $|R|$  from  $\text{id}_{|Q|}$  to  $(\phi_Q^{-1} \circ f \circ \phi_P)^{op} \circ (\phi_P^{-1} \circ g \circ \phi_Q)^{op}$ .

The proof of (ii) is analogous to the one of (i). The details are left to the reader.  $\square$

#### 4. CONSTRUCTION OF d-MAPS

The purpose of this section is to present the method we will use in the next sections to construct d-maps and dihomotopies between geometric realizations of precubical sets.

**Definition 4.1.** A subset  $Z$  of a partially ordered set  $(J, \leq)$  is called *order-convex* if for any two elements  $a, b \in Z$ ,  $\{z \in J \mid a \leq z \leq b\} \subseteq Z$ .

**Remark 4.2.** If  $\alpha: I \rightarrow I^m$  is an order-preserving map and  $s \leq t$  are elements of  $I$  such that  $\alpha(s)$  and  $\alpha(t)$  belong to an order-convex set  $Z \subseteq I^m$ , then  $[s, t] \subseteq \alpha^{-1}(Z)$ .

**Proposition 4.3.** *Let  $P$  and  $Q$  be precubical sets and  $f: \coprod_{r \geq 0} P_r \times I^r \rightarrow |Q|$  be a continuous map. Suppose that*

- (i)  $f(d_i^k z, u) = f(z, \delta_i^k u)$  for all  $r \geq 1$ ,  $z \in P_r$ ,  $u \in I^{r-1}$ ,  $i \in \{1, \dots, r\}$ ,  $k \in \{0, 1\}$ ,
- (ii) for all  $r \geq 1$  and  $z \in P_r$  there exist a finite closed order-convex covering  $\mathcal{A}_z$  of  $I^r$ , a function  $\zeta_z: \mathcal{A}_z \rightarrow \coprod_{m \geq 0} Q_m$ , and a family of order-preserving maps  $\{f_{z,Z}: Z \rightarrow I^{|\zeta_z(Z)|}\}_{Z \in \mathcal{A}_z}$  such that for all  $Z \in \mathcal{A}_z$  and  $u \in Z$ ,  $f(z, u) = [\zeta_z(Z), f_{z,Z}(u)]$ .

Then a  $d$ -map  $\bar{f}: |P| \rightarrow |Q|$  is given by  $\bar{f}([z, u]) = f(z, u)$ .

*Proof.* By condition (i),  $\bar{f}$  is well-defined and continuous. Let  $r \geq 0$  and  $z \in P_r$  and consider a path  $\omega \in d|P|$  of the form  $\omega(t) = [z, \alpha(t)]$  where the map  $\alpha: I \rightarrow I^r$  is order-preserving. We have to show that  $\bar{f} \circ \omega \in d|Q|$ . If  $r = 0$ , then  $\bar{f} \circ \omega$  is a constant path and therefore  $\bar{f} \circ \omega \in d|Q|$ . Let  $r \geq 1$ . Let  $\mathcal{B}_z$  be the subset of  $\mathcal{A}_z$  consisting of the sets  $Z \in \mathcal{A}_z$  such that  $\alpha^{-1}(Z)$  has more than one element. Then  $I = \bigcup_{Z \in \mathcal{B}_z} \alpha^{-1}(Z)$ . Indeed, else there would exist an element  $s \in I \setminus \bigcup_{Z \in \mathcal{B}_z} \alpha^{-1}(Z)$  and one would have  $\bigcup_{\substack{Z \in \mathcal{A}_z \\ s \in \alpha^{-1}(Z)}} \alpha^{-1}(Z) = \{s\}$  and hence  $I \setminus \{s\} = \bigcup_{\substack{Z \in \mathcal{A}_z \\ s \notin \alpha^{-1}(Z)}} \alpha^{-1}(Z)$  which

is impossible since  $I \setminus \{s\}$  is not closed in  $I$ . Define a subset  $\{Z_1, \dots, Z_l\} \subseteq \mathcal{B}_z$  such that  $0 < \max \alpha^{-1}(Z_1) < \dots < \max \alpha^{-1}(Z_l) = 1$ ,  $[0, \max \alpha^{-1}(Z_1)] = \alpha^{-1}(Z_1)$ , and  $[\max \alpha^{-1}(Z_{i-1}), \max \alpha^{-1}(Z_i)] \subseteq \alpha^{-1}(Z_i)$  for  $i \in \{2, \dots, l\}$  inductively as follows. Choose  $Z_1 \in \mathcal{B}_z$  such that  $0 \in \alpha^{-1}(Z_1)$ . Then  $0 < \max \alpha^{-1}(Z_1)$  and, by 4.2,  $[0, \max \alpha^{-1}(Z_1)] = \alpha^{-1}(Z_1)$ . Suppose that  $Z_i$  has been defined and that  $\max \alpha^{-1}(Z_i) < 1$ . Then

$$[0, \max \alpha^{-1}(Z_i)] = \bigcup_{\substack{Z \in \mathcal{B}_z \\ \max \alpha^{-1}(Z) \leq \max \alpha^{-1}(Z_i)}} \alpha^{-1}(Z)$$

and hence

$$[\max \alpha^{-1}(Z_i), 1] \subseteq \bigcup_{\substack{Z \in \mathcal{B}_z \\ \max \alpha^{-1}(Z) > \max \alpha^{-1}(Z_i)}} \alpha^{-1}(Z).$$

Since this is a finite union of closed subsets of  $I$ , it even contains the closed interval  $[\max \alpha^{-1}(Z_i), 1]$  as a subset. Therefore we may choose  $Z_{i+1} \in \mathcal{B}_z$  such that  $\max \alpha^{-1}(Z_i) < \max \alpha^{-1}(Z_{i+1})$  and  $\max \alpha^{-1}(Z_i) \in \alpha^{-1}(Z_{i+1})$ . By 4.2, we then also have  $[\max \alpha^{-1}(Z_i), \max \alpha^{-1}(Z_{i+1})] \subseteq \alpha^{-1}(Z_{i+1})$ . Since  $\mathcal{B}_z$  is finite, the process terminates after a finite number of steps. Set  $b_i = \max \alpha^{-1}(Z_i)$  ( $i = 1, \dots, l$ ) and  $b_0 = 0$ . By 2.6, it suffices to show that for each  $i \in \{1, \dots, l\}$ , the path  $\gamma_i: I \rightarrow |Q|$ ,  $t \mapsto \bar{f} \circ \omega((1-t)b_{i-1} + tb_i)$  belongs to  $d|Q|$ . Let  $\beta_i$  be the composite

$$I \xrightarrow{(1-t)b_{i-1} + tb_i} [b_{i-1}, b_i] \xrightarrow{\alpha} Z_i \xrightarrow{f_{z,Z_i}} I^{|\zeta_z(Z_i)|}.$$

Then  $\beta_i$  is order-preserving and  $\gamma_i(t) = [\zeta_z(Z_i), \beta_i(t)]$  ( $t \in I$ ). Thus,  $\gamma_i \in d|Q|$ .  $\square$

**Proposition 4.4.** *Let  $P$  and  $Q$  be precubical sets and  $h: \coprod_{r \geq 0} P_r \times I^r \times I \rightarrow |Q|$  be a continuous map. Suppose that*

- (i)  $h(d_i^k z, u, t) = h(z, \delta_i^k u, t)$  for all  $r \geq 1$ ,  $z \in P_r$ ,  $u \in I^{r-1}$ ,  $i \in \{1, \dots, r\}$ ,  $k \in \{0, 1\}$ ,  $t \in I$ ,
- (ii) for all  $t \in I$ ,  $r \geq 1$ , and  $z \in P_r$  there exist a finite closed order-convex covering  $\mathcal{A}_{z,t}$  of  $I^r$ , a function  $\zeta_{z,t}: \mathcal{A}_{z,t} \rightarrow \coprod_{m \geq 0} Q_m$ , and a family of order-preserving maps  $\{h_{z,t,Z}: Z \rightarrow I^{|\zeta_{z,t}(Z)|}\}_{Z \in \mathcal{A}_{z,t}}$  such that for all  $Z \in \mathcal{A}_{z,t}$  and  $u \in Z$ ,  $h(z, u, t) = [\zeta_{z,t}(Z), h_{z,t,Z}(u)]$ .

Then a dihomotopy  $H: |P| \times I \rightarrow |Q|$  is given by  $H([z, u], t) = h(z, u, t)$ .

*Proof.* By condition (i),  $H$  is well-defined and continuous. By 4.3, the map  $H(\cdot, t): |P| \rightarrow |Q|$  is a d-map for each  $t \in I$ .  $\square$

## 5. ONE-DIMENSIONAL REDUCTION

In ordinary homotopy theory, a contractible subspace of a topological space can be collapsed to a point, at least if the space and the subspace form an NDR-pair. The resulting quotient space has the same homotopy type as the original space. In directed homotopy theory, the situation is more complicated. Consider, for example, the geometric realization of a precubical set  $P$  with two vertices and two edges that looks like the graph in figure 1(b). If one collapses one of the edges to a point, one obtains the directed circle  $\vec{S}^1$ , which is the geometric realization of a precubical set with one vertex and one edge. The d-spaces  $|P|$  and  $\vec{S}^1$  are *not* dihomotopy equivalent. The following theorem gives a condition under which it is possible to collapse an edge in the geometric realization of a precubical set to a point without changing the directed homotopy type relative to the extremal elements:

**Theorem 5.1.** *Let  $P$  be a precubical set,  $b \in \{0, 1\}$ , and  $x \in P_1$  be a regular element such that*

- (i) *there is no element  $y \in P_1 \setminus \{x\}$  such that  $d_1^{1-b}y = d_1^{1-b}x$ ,*
- (ii) *no element in  $P_1$  having  $d_1^{1-b}x$  in its boundary belongs to the boundary of an element in  $P_2$ .*

*Consider the set  $Y = \{y \in P_1 \mid d_1^b y = d_1^{1-b}x\}$ . Then a precubical set  $Q$  such that  $Y \subseteq Q_1$ ,  $Q \setminus Y$  is a common precubical subset of  $P$  and  $Q$ , and  $|P|$  and  $|Q|$  are dihomotopy equivalent relative to  $|Q \setminus Y|$  is given by  $Q_0 = P_0 \setminus \{d_1^{1-b}x\}$ ,  $Q_1 = P_1 \setminus \{x\}$ ,  $Q_r = P_r$  ( $r > 1$ ), and the boundary operators  $D_i^k$  defined by*

$$D_i^k z = \begin{cases} d_1^b x, & z \in Y, i = 1, k = b, \\ d_i^k z, & \text{else.} \end{cases}$$

*Moreover, if  $Y \neq \emptyset$ , then  $\text{Extrl}(P) = \text{Extrl}(Q) \subseteq Q \setminus Y$  and  $|P|$  and  $|Q|$  have isomorphic fundamental bipartite graphs.*

*Proof.* We first consider the case  $b = 0$ . Since  $x$  is regular,  $d_1^0 x \neq d_1^1 x$ . Therefore  $x \notin Y$  and  $Y \subseteq Q_1$ . We check that the boundary operators of  $Q$  are well-defined. By condition (ii),  $x$  is not in the boundary of any element of  $P_2$ . Hence  $x \neq d_i^k z = D_i^k z$  for all  $z \in Q_2$  and all  $i$  and  $k$ . For all  $y \in Y$ ,  $D_1^0 y = d_1^0 x \neq d_1^1 x$ . For  $z \in Q_1 \setminus Y$ ,  $D_1^0 z = d_1^0 z \neq d_1^1 x$  by definition of  $Y$ . By condition (i),  $D_1^1 z = d_1^1 z \neq d_1^1 x$  for all  $z \in Q_1$ . It follows that the boundary operators of  $Q$  are well-defined. Let  $r \geq 2$  and  $z \in Q_r = P_r$ . By condition (ii),  $d_i^k z \notin Y$  for all  $i$  and  $k$ . Therefore  $D_i^k D_j^l z = D_i^k d_j^l z = d_i^k d_j^l z = d_{j-1}^l d_i^k z = d_{j-1}^l D_i^k z = D_{j-1}^l D_i^k z$  whenever  $i < j$ .

It follows that  $Q$  is a precubical set. Since, by condition (ii), no element of  $Y$  is in the boundary of a higher dimensional element of  $Q$ ,  $Q \setminus Y$  is a precubical subset of  $Q$ . By condition (ii), no element of  $Y \cup \{x\}$  is in the boundary of a higher dimensional element of  $P$ . Since, by condition (i) and the definition of  $Y$ , no element of  $P \setminus (Y \cup \{x\})$  has  $d_1^1 x$  in its boundary,  $P \setminus (Y \cup \{x, d_1^1 x\})$  is a precubical subset of  $P$ . We have  $Q \setminus Y = P \setminus (Y \cup \{x, d_1^1 x\})$ , and  $Q \setminus Y$  is a common precubical subset of  $P$  and  $Q$ .

Consider the continuous map  $f: \coprod_{r \geq 0} P_r \times I^r \rightarrow |Q|$  defined by

$$f(z, u) = \begin{cases} [d_1^0 x, ()], & z \in \{x, d_1^1 x\}, \\ [z, u], & z \notin \{x, d_1^1 x\}. \end{cases}$$

We show that  $f$  satisfies the conditions of 4.3. Consider  $r \geq 1$ ,  $z \in P_r$ ,  $u \in I^{r-1}$ ,  $i \in \{1, \dots, r\}$ , and  $k \in \{0, 1\}$ . We show that  $f(z, \delta_i^k u) = f(d_i^k z, u)$ . Suppose first that  $z \in \{x, d_1^1 x\}$ . Then  $z = x$ ,  $r = i = 1$ , and  $u = ()$ . We have  $f(d_1^1 x, ()) = [d_1^0 x, ()] = f(x, 1) = f(x, \delta_1^1 ())$  and  $f(d_1^0 x, ()) = [d_1^0 x, ()] = f(x, 0) = f(x, \delta_1^0 ())$ . Suppose now that  $z \notin \{x, d_1^1 x\}$  but  $d_i^k z \in \{x, d_1^1 x\}$ . Then  $d_i^k z = d_1^1 x$ . By condition (i) and the definition of  $Y$ ,  $r = i = 1$ ,  $k = 0$ ,  $z \in Y$ , and  $u = ()$ . We have  $f(z, \delta_1^0 ()) = f(z, 0) = [z, 0] = [D_1^0 z, ()] = [d_1^0 x, ()] = f(d_1^0 z, ()) = f(d_1^0 z, 0)$ . Suppose finally that  $z \notin \{x, d_1^1 x\}$  and  $d_i^k z \notin \{x, d_1^1 x\}$ . Then  $z \notin Y$  or  $(i, k) \neq (1, 0)$  because otherwise one would have  $d_i^k z = d_1^0 z = d_1^1 x$ . Thus,  $D_i^k z = d_i^k z$  and  $f(z, \delta_i^k u) = [z, \delta_i^k u] = [D_i^k z, u] = [d_i^k z, u] = f(d_i^k z, u)$ . It follows that  $f$  satisfies condition 4.3 (i). For condition 4.3 (ii) let  $r \geq 1$  and  $z \in P_r$ . Set  $\mathcal{A}_z = \{I^r\}$ ,  $\zeta_z(I^r) = d_1^0 x$ , and  $f_{z, I^r}(u) = ()$  if  $z \in \{x, d_1^1 x\}$  and  $\mathcal{A}_z = \{I^r\}$ ,  $\zeta_z(I^r) = z$ , and  $f_{z, I^r}(u) = u$  if  $z \notin \{x, d_1^1 x\}$ . Then  $f_{z, I^r}$  is order-preserving and we have  $f(z, u) = [\zeta_z(I^r), f_{z, I^r}(u)]$  for all  $u \in I^r$ . It follows that  $f$  satisfies both conditions of 4.3 and that the map  $\bar{f}: |P| \rightarrow |Q|$ ,  $\bar{f}([z, u]) = f(z, u)$  is a d-map.

Consider the continuous map  $g: \coprod_{r \geq 0} Q_r \times I^r \rightarrow |P|$  defined by

$$g(z, u) = \begin{cases} [x, 2u], & z \in Y, u \leq 1/2 \\ [z, 2u - 1], & z \in Y, u \geq 1/2 \\ [z, u], & z \notin Y. \end{cases}$$

We show that  $g$  satisfies the conditions of 4.3. Consider  $r \geq 1$ ,  $z \in Q_r$ ,  $u \in I^{r-1}$ ,  $i \in \{1, \dots, r\}$ , and  $k \in \{0, 1\}$ . We show that  $g(z, \delta_i^k u) = g(D_i^k z, u)$ . Suppose first that  $z \in Y$ . Then  $r = i = 1$  and  $u = ()$ . We have  $g(z, \delta_1^0 ()) = g(z, 0) = [x, 0] = [d_1^0 x, ()] = [D_1^0 z, ()] = g(D_1^0 z, ())$  and  $g(z, \delta_1^1 ()) = g(z, 1) = [z, 1] = [d_1^1 z, ()] = [D_1^1 z, ()] = g(D_1^1 z, ())$ . Suppose now that  $z \notin Y$ . Then  $D_i^k z = d_i^k z \notin Y$  and  $g(z, \delta_i^k u) = [z, \delta_i^k u] = [d_i^k z, u] = [D_i^k z, u] = g(D_i^k z, u)$ . It follows that  $g$  satisfies condition 4.3 (i). For condition 4.3 (ii) let  $r \geq 1$  and  $z \in Q_r$ . Set  $\mathcal{A}_z = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ ,  $\zeta_z([0, \frac{1}{2}]) = x$ ,  $\zeta_z([\frac{1}{2}, 1]) = z$ ,  $g_{z, [0, \frac{1}{2}]}(u) = 2u$ , and  $g_{z, [\frac{1}{2}, 1]}(u) = 2u - 1$  if  $z \in Y$  and  $\mathcal{A}_z = \{I^r\}$ ,  $\zeta_z(I^r) = z$ , and  $g_{z, I^r}(u) = u$  if  $z \notin Y$ . Then  $g_{z, Z}$  is order-preserving for each  $Z \in \mathcal{A}_z$  and we have  $g(z, u) = [\zeta_z(Z), g_{z, Z}(u)]$  for all  $Z \in \mathcal{A}_z$  and  $u \in Z$ . It follows that  $g$  satisfies both conditions of 4.3 and that the map  $\bar{g}: |Q| \rightarrow |P|$ ,  $\bar{g}([z, u]) = g(z, u)$  is a d-map.

Note that both  $\bar{f}$  and  $\bar{g}$  restrict to the identity on  $|Q \setminus Y| = |P \setminus (Y \cup \{x, d_1^1 x\})|$ . We show that  $\bar{f}$  and  $\bar{g}$  are inverse dihomotopy equivalences relative to  $|Q \setminus Y|$ .

Consider the map  $\phi: \coprod_{r \geq 0} P_r \times I^r \times I \rightarrow |P|$  defined by

$$\phi(z, u, t) = \begin{cases} [z, u], & z \in \{x, d_1^1 x\}, t \leq 1/2 \\ [x, (2-2t)u], & z = x, t \geq 1/2, \\ [x, 2-2t], & z = d_1^1 x, t \geq 1/2, \\ [z, (1-2t)u], & z \in Y, u \leq 1/2, t \leq 1/2, \\ [x, 2-2t + (2t-1)2u], & z \in Y, u \leq 1/2, t \geq 1/2, \\ [z, (1-2t)u + 2t(2u-1)], & z \in Y, u \geq 1/2, t \leq 1/2, \\ [z, 2u-1], & z \in Y, u \geq 1/2, t \geq 1/2, \\ [z, u], & z \notin Y \cup \{x, d_1^1 x\}. \end{cases}$$

It is straightforward to check that  $\phi$  is well-defined and continuous. We show that  $\phi$  satisfies the conditions of 4.4. Consider  $r \geq 1$ ,  $z \in P_r$ ,  $u \in I^{r-1}$ ,  $t \in I$ ,  $i \in \{1, \dots, r\}$ , and  $k \in \{0, 1\}$ . We show that  $\phi(z, \delta_i^k u, t) = \phi(d_i^k z, u, t)$ . Suppose first that  $z \in \{x, d_1^1 x\}$ . Then  $z = x$ ,  $r = i = 1$ , and  $u = ()$ . If  $t \leq \frac{1}{2}$ , then  $\phi(d_1^1 x, (), t) = [d_1^1 x, ()] = [x, 1] = \phi(x, 1, t) = \phi(x, \delta_1^1(), t)$ . If  $t \geq \frac{1}{2}$ , then  $\phi(d_1^1 x, (), t) = [x, 2-2t] = \phi(x, 1, t) = \phi(x, \delta_1^1(), t)$ . For all  $t \in I$ ,  $\phi(d_1^0 x, (), t) = [d_1^0 x, ()] = [x, 0] = \phi(x, 0, t) = \phi(x, \delta_1^0(), t)$ . Suppose now that  $z \in Y$ . Then  $r = i = 1$  and  $u = ()$ . For  $t \leq \frac{1}{2}$  we have  $\phi(d_1^0 z, (), t) = \phi(d_1^1 x, (), t) = [d_1^1 x, ()] = [d_1^0 z, ()] = [z, 0] = \phi(z, 0, t) = \phi(z, \delta_1^0(), t)$ . For  $t \geq \frac{1}{2}$  we have  $\phi(d_1^0 z, (), t) = \phi(d_1^1 x, (), t) = [x, 2-2t] = \phi(z, 0, t) = \phi(z, \delta_1^0(), t)$ . For all  $t \in I$ ,  $\phi(d_1^1 z, (), t) = [d_1^1 z, ()] = [z, 1] = \phi(z, 1, t) = \phi(z, \delta_1^1(), t)$ . Suppose finally that  $z \notin Y \cup \{x, d_1^1 x\}$ . Then also  $d_i^k z \notin Y \cup \{x, d_1^1 x\}$ . Thus,  $\phi(z, \delta_i^k u, t) = [z, \delta_i^k u] = [d_i^k z, u] = \phi(d_i^k z, u, t)$ . It follows that  $\phi$  satisfies condition 4.4 (i). We check condition 4.4 (ii). For all  $t \in I$  set  $\mathcal{A}_{x,t} = \{I\}$  and  $\zeta_{x,t}(I) = x$ . For  $t \leq \frac{1}{2}$  set  $\phi_{x,t,I}(u) = u$  and for  $t > \frac{1}{2}$  set  $\phi_{x,t,I}(u) = (2-2t)u$ . For  $z \in P_{\geq 1} \setminus (Y \cup \{x\})$  and  $t \in I$  set  $\mathcal{A}_{z,t} = \{I^{|z|}\}$ ,  $\zeta_{z,t}(I^{|z|}) = z$ , and  $\phi_{z,t,I^{|z|}}(u) = u$ . For  $z \in Y$  and  $t \leq \frac{1}{2}$  set  $\mathcal{A}_{z,t} = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ ,  $\zeta_{z,t}([0, \frac{1}{2}]) = z$ ,  $\zeta_{z,t}([\frac{1}{2}, 1]) = z$ ,  $\phi_{z,t,[0, \frac{1}{2}]}(u) = (1-2t)u$ , and  $\phi_{z,t,[\frac{1}{2}, 1]}(u) = (1-2t)u + 2t(2u-1)$ . For  $z \in Y$  and  $t > \frac{1}{2}$  set  $\mathcal{A}_{z,t} = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ ,  $\zeta_{z,t}([0, \frac{1}{2}]) = x$ ,  $\zeta_{z,t}([\frac{1}{2}, 1]) = z$ ,  $\phi_{z,t,[0, \frac{1}{2}]}(u) = 2-2t + (2t-1)2u$ , and  $\phi_{z,t,[\frac{1}{2}, 1]}(u) = 2u-1$ . Then for all  $z \in P_{\geq 1}$  and  $t \in I$ ,  $\mathcal{A}_{z,t}$  is a closed order-convex covering of  $I^{|z|}$ . Moreover, for all  $Z \in \mathcal{A}_{z,t}$ ,  $\phi_{z,t,Z}$  is order-preserving and we have  $\phi(z, u, t) = [\zeta_{z,t}(Z), \phi_{z,t,Z}(u)]$  for all  $u \in Z$ . It follows that  $\phi$  satisfies both conditions of 4.4 and that the map  $\Phi: |P| \times I \rightarrow |P|$ ,  $\Phi([z, u], t) = \phi(z, u, t)$  is a dihomotopy. We have  $\Phi([z, u], 0) = [z, u]$  and  $\Phi([z, u], 1) = \bar{g} \circ \bar{f}([z, u])$ . Moreover,  $\Phi([z, u], t) = [z, u]$  for all  $[z, u] \in |P| \setminus (Y \cup \{x, d_1^1 x\})$  and  $t \in I$ . It follows that  $\Phi$  is a dihomotopy relative to  $|P| \setminus (Y \cup \{x, d_1^1 x\}) = |Q| \setminus Y$  from  $id_{|P|}$  to  $\bar{g} \circ \bar{f}$ .

Consider the continuous map  $\psi: \coprod_{r \geq 0} Q_r \times I^r \times I \rightarrow |Q|$  defined by

$$\psi(z, u, t) = \begin{cases} [z, (1-t)u], & z \in Y, u \leq 1/2, \\ [z, (1-t)u + t(2u-1)], & z \in Y, u \geq 1/2, \\ [z, u], & z \notin Y. \end{cases}$$

We show that  $\psi$  satisfies the conditions of 4.4. Consider  $r \geq 1$ ,  $z \in Q_r$ ,  $u \in I^{r-1}$ ,  $t \in I$ ,  $i \in \{1, \dots, r\}$ , and  $k \in \{0, 1\}$ . We show that  $\psi(z, \delta_i^k u, t) = \psi(D_i^k z, u, t)$ . Suppose first that  $z \in Y$ . Then  $r = i = 1$  and  $u = ()$ . We have  $\psi(z, \delta_1^0(), t) = \psi(z, 0, t) = [z, 0] = [D_1^0 z, ()] = \psi(D_1^0 z, (), t)$  and  $\psi(z, \delta_1^1(), t) = \psi(z, 1, t) = [z, 1] = [D_1^1 z, ()] = \psi(D_1^1 z, (), t)$ . Suppose now that  $z \notin Y$ . Then  $D_i^k z \notin Y$  and  $\psi(z, \delta_i^k u, t) = [z, \delta_i^k u] =$

$[D_i^k z, u] = \psi(D_i^k z, u, t)$ . It follows that  $\psi$  satisfies condition (i) of 4.4. We check condition 4.4 (ii). For  $z \in Y$  and  $t \in I$  set  $\mathcal{A}_{z,t} = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ ,  $\zeta_{z,t}([0, \frac{1}{2}]) = z$ ,  $\zeta_{z,t}([\frac{1}{2}, 1]) = z$ ,  $\psi_{z,t,[0, \frac{1}{2}]}(u) = (1-t)u$ , and  $\psi_{z,t,[\frac{1}{2}, 1]}(u) = (1-t)u + t(2u-1)$ . For  $z \in Q_{\geq 1} \setminus Y$  and  $t \in I$  set  $\mathcal{A}_{z,t} = \{I^{|z|}\}$ ,  $\zeta_{z,t}(I^{|z|}) = z$ , and  $\psi_{z,t,I^{|z|}}(u) = u$ . Then for all  $z \in Q_{\geq 1}$  and  $t \in I$ ,  $\mathcal{A}_{z,t}$  is a closed order-convex covering of  $I^{|z|}$ . Moreover, for all  $Z \in \mathcal{A}_{z,t}$ ,  $\psi_{z,t,Z}$  is order-preserving and we have  $\psi(z, u, t) = [\zeta_{z,t}(Z), \psi_{z,t,Z}(u)]$  for all  $u \in Z$ . It follows that  $\psi$  satisfies both conditions of 4.4 and that the map  $\Psi: |Q| \times I \rightarrow |Q|$ ,  $\Psi([z, u], t) = \psi(z, u, t)$  is a dihomotopy. We have  $\Psi([z, u], 0) = [z, u]$  and  $\Psi([z, u], 1) = \bar{f} \circ \bar{g}([z, u])$ . Moreover,  $\Psi([z, u], t) = [z, u]$  for all  $[z, u] \in |Q \setminus Y|$  and  $t \in I$ . It follows that  $\Psi$  is a dihomotopy relative to  $|Q \setminus Y|$  from  $id_{|Q|}$  to  $\bar{f} \circ \bar{g}$ .

Suppose now that  $Y \neq \emptyset$ . Let  $v \in P_0 = Q_0 \cup \{d_1^1 x\}$  be a minimal element of  $P$ . Then for all  $z \in P_1$ ,  $d_1^1 z \neq v$ . Thus,  $v \in Q_0$  and for all  $z \in Q_1 = P_1 \setminus \{x\}$ ,  $D_1^1 z = d_1^1 z \neq v$ . Hence  $v$  is a minimal element of  $Q$ . Let  $v \in P_0 = Q_0 \cup \{d_1^1 x\}$  be a maximal element of  $P$ . Then for all  $z \in P_1$ ,  $d_1^0 z \neq v$ . Since  $Y \neq \emptyset$ ,  $v \neq d_1^1 x$  and  $v \in Q_0$ . For all  $z \in Q_1$ , we have  $D_1^0 z = d_1^0 z'$  for some  $z' \in P_1$  and hence  $D_1^0 z \neq v$ . Therefore  $v$  is a maximal element of  $Q$ . It follows that  $Extrl(P) \subseteq Extrl(Q)$ . Let  $v \in Q_0 = P_0 \setminus \{d_1^1 x\}$  be a minimal element of  $Q$ . Then  $v \in P_0$ ,  $d_1^1 x \neq v$ , and for all  $z \in Q_1 = P_1 \setminus \{x\}$ ,  $d_1^1 z = D_1^1 z \neq v$ . Hence  $v$  is a minimal element of  $P$ . Let  $v \in Q_0 = P_0 \setminus \{d_1^1 x\}$  be a maximal element of  $Q$ . Then  $v \in P_0$  and for all  $z \in Q_1 = P_1 \setminus \{x\}$ ,  $D_1^0 z \neq v$ . Let  $y \in Y \neq \emptyset$ . Since  $Y \subseteq Q_1$ ,  $d_1^0 x = D_1^0 y \neq v$ . For  $z \in Y$ ,  $d_1^0 z = d_1^1 x \neq v$ . For  $z \in P_1 \setminus (Y \cup \{x\})$ ,  $d_1^0 z = D_1^0 z \neq v$ . It follows that  $d_1^0 z \neq v$  for all  $z \in P_1$  and hence that  $v$  is maximal in  $P$ . We therefore have that  $Extrl(Q) \subseteq Extrl(P)$  and hence that  $Extrl(P) = Extrl(Q) \subseteq Q \setminus Y$ . By 3.7, this implies that  $|P|$  and  $|Q|$  have isomorphic fundamental bipartite graphs. This terminates the proof of the theorem in the case  $b = 0$ .

Suppose now that  $b = 1$ . Then  $P^{op}$  satisfies the conditions of the theorem in the case  $b = 0$ . It follows that  $Y \subseteq Q_1$  and that  $Q$  is a precubical set with respect to the boundary operators  $\tilde{D}_i^k$  defined by

$$\tilde{D}_i^k z = \begin{cases} d_1^1 x, & z \in Y, i = 1, k = 0, \\ d_i^{1-k} z, & \text{else.} \end{cases}$$

The opposite precubical set is  $Q$  with the boundary operators  $D_i^k$ . We denote this precubical set by  $Q$ , so that  $Q^{op}$  is the graded set  $Q$  with the boundary operators  $\tilde{D}_i^k$ . By the theorem in the case  $b = 0$ ,  $Q^{op} \setminus Y$  is a common precubical subset of  $Q^{op}$  and  $P^{op}$  and  $|P^{op}|$  and  $|Q^{op}|$  are dihomotopy equivalent relative to  $|Q^{op} \setminus Y|$ . It follows that  $Q \setminus Y = (Q^{op} \setminus Y)^{op}$  is a common precubical subset of  $P$  and  $Q$  and, by 3.8(i), that  $|P|$  and  $|Q|$  are dihomotopy equivalent relative to  $|Q \setminus Y|$ . If  $Y \neq \emptyset$ , then by the theorem in the case  $b = 0$ ,  $Extrl(P^{op}) = Extrl(Q^{op}) \subseteq Q^{op} \setminus Y$ . It follows that  $Extrl(P) = Extrl(Q) \subseteq Q \setminus Y$  and hence, by 3.7, that  $|P|$  and  $|Q|$  have isomorphic fundamental bipartite graphs.  $\square$

## 6. TWO-DIMENSIONAL REDUCTION

In opposition to the situation in ordinary homotopy theory, the removal of a cube and a free face from a precubical set changes in general the directed homotopy type of the geometric realization. In this section we prove two theorems which give conditions under which it is possible to eliminate a 2-dimensional cube and one

or two free faces in a precubical set without changing the directed homotopy type relative to the extremal elements of the geometric realization.

**Theorem 6.1.** *Let  $P$  be a precubical set,  $a \in \{1, 2\}$ ,  $b \in \{0, 1\}$ , and  $x \in P_2$  be a regular element such that*

- (i) *no element of  $P_2 \setminus \{x\}$  has  $d_a^{1-b}x$  or  $d_{3-a}^b x$  in its boundary,*
- (ii) *there is no element  $y \in P_1 \setminus \{d_a^{1-b}x\}$  such that  $d_1^b y = d_1^b d_a^{1-b}x$ ,*
- (iii) *no element of the set  $Y = \{y \in P_1 \setminus \{d_{3-a}^b x\} \mid d_1^{1-b}y = d_1^{1-b}d_{3-a}^b x\}$  is in the boundary of some element in  $P_2$ .*

*Then a precubical subset  $Q$  of  $P$  and a precubical subset  $R$  of  $Q$  such that the inclusion  $\iota: |Q| \hookrightarrow |P|$  is a dihomotopy equivalence relative to  $|R|$  are given by  $Q_0 = P_0$ ,  $Q_1 = P_1 \setminus \{d_{3-a}^b x\}$ ,  $Q_2 = P_2 \setminus \{x\}$ ,  $Q_r = P_r$  ( $r > 2$ ),  $R_0 = Q_0 \setminus \{d_1^{1-b}d_{3-a}^b x\}$ ,  $R_1 = Q_1 \setminus (\{d_a^{1-b}x\} \cup Y)$ , and  $R_r = Q_r$  ( $r \geq 2$ ). Moreover, if  $Y \neq \emptyset$ , then  $\text{Extrl}(P) = \text{Extrl}(Q) \subseteq R$  and  $\iota$  induces an isomorphism of fundamental bipartite graphs  $\pi_1(|Q|, \text{Extrl}(|Q|)) \xrightarrow{\cong} \pi_1(|P|, \text{Extrl}(|P|))$ .*

*Proof.* We first consider the case  $a = 1$  and  $b = 0$ . In order to see that  $Q$  is a precubical subset of  $P$ , one has to check that it is stable under the boundary operators. Condition (i) implies that this is the case in degrees  $\leq 2$ . In degrees  $> 2$ , one has to check that  $x$  is not in the boundary of any element  $z \in P_3$ . If such an element existed, there would exist  $j \in \{1, 2, 3\}$  and  $l \in \{0, 1\}$  such that  $d_j^l z = x$ . It is impossible that  $j = 1$  or  $(j, l) = (2, 1)$  because otherwise one would have  $d_j^l d_3^0 z = d_2^0 d_j^l z = d_2^0 x$  and consequently  $d_3^0 z = x$ . But this would imply that  $d_j^l x = d_2^0 x$  and hence that  $x$  is not regular. Similarly, it is also impossible that  $j = 3$  or  $(j, l) = (2, 0)$  because otherwise one would have  $d_{j-1}^l d_1^1 z = d_1^1 d_j^l z = d_1^1 x$  and consequently  $d_1^1 z = x$ . This would imply that  $d_{j-1}^l x = d_1^1 x$  and hence that  $x$  is not regular. It follows that  $x$  is not in the boundary of any element of  $P_3$  and that  $Q$  is a precubical subset of  $P$ . By condition (ii), the set of elements of  $Q_1$  having  $d_1^1 d_2^0 x$  in their boundary is  $\{d_1^1 x\} \cup Y$ . Therefore no element of  $R_1$  has  $d_1^1 d_2^0 x$  in its boundary. By conditions (i) and (iii), no element of  $\{d_1^1 x\} \cup Y$  is in the boundary of an element of  $R_2$ . It follows that  $R$  is a precubical subset of  $Q$ .

We construct a dihomotopy  $H: |P| \times I \rightarrow |P|$  using 4.4. Consider the map  $h: \coprod_{r \geq 0} P_r \times I^r \times I \rightarrow |P|$  defined by

$$h(x, (u_1, u_2), t)$$

$$= \begin{cases} [x, (u_1, u_2)], & t \leq \frac{1}{3}, \\ [x, (u_1, u_2)], & \frac{1}{3} \leq t \leq \frac{2}{3}, u_1 \leq u_2, \\ [x, (u_1, (3t-1)u_1 + (2-3t)u_2)], & \frac{1}{3} \leq t \leq \frac{2}{3}, u_1 \geq u_2, \\ [x, ((3-3t)u_1, u_2 + (3t-2)u_1)], & t \geq \frac{2}{3}, u_1 \leq u_2, \\ [x, (u_1 + (3t-2)(u_2-1), (3-3t)u_2 + 3t-2)], & t \geq \frac{2}{3}, u_1 \leq u_2, \\ [x, ((3-3t)u_1, (3t-1)u_1)], & u_2 \geq 1-u_1, \\ [x, ((3t-1)(u_1-1) + 1, (3-3t)u_1 + 3t-2)], & t \geq \frac{2}{3}, u_1 \geq u_2, u_1 \leq \frac{1}{2}, \\ & t \geq \frac{2}{3}, u_1 \geq u_2, u_1 \geq \frac{1}{2}, \end{cases}$$

$$h(d_1^1 x, u, t) = h(x, (1, u), t), \quad h(d_2^0 x, u, t) = h(x, (u, 0), t), \quad h(d_1^1 d_2^0 x, (), t) = h(x, (1, 0), t),$$

$$h(y, u, t) = \begin{cases} [y, (1+3t)u], & t \leq \frac{1}{3}, u \leq \frac{1}{2}, \\ [y, (1-3t)u + 3t], & t \leq \frac{1}{3}, u \geq \frac{1}{2}, \\ [y, 2u], & t \geq \frac{1}{3}, u \leq \frac{1}{2}, \\ [d_1^1 x, (3t-1)(2u-1)], & \frac{1}{3} \leq t \leq \frac{2}{3}, u \geq \frac{1}{2}, \\ [d_1^1 x, 2u-1], & t \geq \frac{2}{3}, u \geq \frac{1}{2} \end{cases}$$

for  $y \in Y$ , and  $h(z, (u_1, \dots, u_r), t) = [z, (u_1, \dots, u_r)]$  for  $z \in R$ . It is straightforward to check that  $h$  is well-defined and continuous. It is also straightforward to check that condition 4.4(i) is satisfied.

We check condition 4.4(ii). For  $t \leq \frac{2}{3}$  we set  $\mathcal{A}_{x,t} = \{I^2\}$  and  $\zeta_{x,t}(I^2) = x$ . We set  $h_{x,t,I^2} = id_{I^2}$  for  $t \leq \frac{1}{3}$  and

$$h_{x,t,I^2}(u_1, u_2) = \begin{cases} (u_1, u_2) & u_1 \leq u_2, \\ (u_1, (3t-1)u_1 + (2-3t)u_2) & u_1 \geq u_2 \end{cases}$$

for  $\frac{1}{3} < t \leq \frac{2}{3}$ . Then  $h_{x,t,I^2}$  is order-preserving for each  $t \leq \frac{2}{3}$  and we have  $h(x, (u_1, u_2), t) = [\zeta_{x,t}(I^2), h_{x,t,I^2}(u_1, u_2)]$  for all  $(u_1, u_2) \in I^2$ . Let  $t > \frac{2}{3}$ . One easily checks that the closed subsets  $Z_1, Z_2 \subseteq I^2$  given by

$$\begin{aligned} Z_1 &= \{(u_1, u_2) \in I^2 \mid u_2 \leq 1 - u_1, u_1 \leq \frac{1}{2}\} \\ &= \{(u_1, u_2) \in I^2 \mid (u_1 \leq u_2, u_2 \leq 1 - u_1) \text{ or } (u_1 \geq u_2, u_1 \leq \frac{1}{2})\} \end{aligned}$$

and

$$\begin{aligned} Z_2 &= \{(u_1, u_2) \in I^2 \mid u_2 \geq 1 - u_1 \text{ or } u_1 \geq \frac{1}{2}\} \\ &= \{(u_1, u_2) \in I^2 \mid (u_1 \leq u_2, u_2 \geq 1 - u_1) \text{ or } (u_1 \geq u_2, u_1 \geq \frac{1}{2})\} \end{aligned}$$

are order-convex and satisfy  $Z_1 \cup Z_2 = I^2$ . We set  $\mathcal{A}_{x,t} = \{Z_1, Z_2\}$ ,  $\zeta_{x,t}(Z_1) = \zeta_{x,t}(Z_2) = x$ ,

$$h_{x,t,Z_1}(u_1, u_2) = \begin{cases} ((3-3t)u_1, u_2 + (3t-2)u_1) & u_1 \leq u_2, u_2 \leq 1 - u_1, \\ ((3-3t)u_1, (3t-1)u_1), & u_1 \geq u_2, u_1 \leq \frac{1}{2}, \end{cases}$$

and

$$h_{x,t,Z_2}(u_1, u_2) = \begin{cases} (u_1 + (3t-2)(u_2-1), (3-3t)u_2 + 3t-2), & u_1 \leq u_2, \\ & u_2 \geq 1 - u_1, \\ ((3t-1)(u_1-1) + 1, (3-3t)u_1 + 3t-2), & u_1 \geq u_2, u_1 \geq \frac{1}{2}. \end{cases}$$

It is straightforward to check that  $h_{x,t,Z_1}$  and  $h_{x,t,Z_2}$  are order-preserving and that  $h(x, (u_1, u_2), t) = [\zeta_{x,t}(Z_i), h_{x,t,Z_i}(u_1, u_2)]$  for all  $(u_1, u_2) \in Z_i$  ( $i = 1, 2$ ).

For each  $t \in I$  we set  $\mathcal{A}_{d_1^1 x, t} = \{I\}$  and  $\zeta_{d_1^1 x, t}(I) = x$ . For  $t \leq \frac{1}{3}$  we set  $h_{d_1^1 x, t, I}(u) = (1, u)$ . For  $\frac{1}{3} < t \leq \frac{2}{3}$  we set  $h_{d_1^1 x, t, I}(u) = (1, (3t-1) + (2-3t)u)$ . For  $t > \frac{2}{3}$  we set  $h_{d_1^1 x, t, I}(u) = (1, 1)$ . Then each map  $h_{d_1^1 x, t, I}$  is order-preserving and satisfies  $h(d_1^1 x, u, t) = [\zeta_{d_1^1 x, t}(I), h_{d_1^1 x, t, I}(u)]$  for all  $u \in I$ .

For  $t \leq \frac{2}{3}$  we set  $\mathcal{A}_{d_2^0 x, t} = \{I\}$  and  $\zeta_{d_2^0 x, t}(I) = x$ . For  $t \leq \frac{1}{3}$  we set  $h_{d_2^0 x, t, I}(u) = (u, 0)$ . For  $\frac{1}{3} < t \leq \frac{2}{3}$  we set  $h_{d_2^0 x, t, I}(u) = (u, (3t-1)u)$ . Then for each  $t \leq \frac{2}{3}$ ,  $h_{d_2^0 x, t, I}$  is order-preserving and satisfies  $h(d_2^0 x, u, t) = [\zeta_{d_2^0 x, t}(I), h_{d_2^0 x, t, I}(u)]$  for all



$u \in I$ . Let  $t > \frac{2}{3}$ . We set  $\mathcal{A}_{d_2^0 x, t} = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ ,  $\zeta_{d_2^0 x, t}([0, \frac{1}{2}]) = \zeta_{d_2^0 x, t}([\frac{1}{2}, 1]) = x$ ,  $h_{d_2^0 x, t, [0, \frac{1}{2}]}(u) = ((3-3t)u, (3t-1)u)$ , and

$$h_{d_2^0 x, t, [\frac{1}{2}, 1]}(u) = ((3t-1)(u-1) + 1, (3-3t)u + 3t - 2).$$

Then  $\mathcal{A}_{d_2^0 x, t}$  is a closed order-convex covering of  $I$  and  $h_{d_2^0 x, t, [0, \frac{1}{2}]}$  and  $h_{d_2^0 x, t, [\frac{1}{2}, 1]}$  are order-preserving. Moreover,  $h(d_2^0 x, u, t) = [\zeta_{d_2^0 x, t}([0, \frac{1}{2}]), h_{d_2^0 x, t, [0, \frac{1}{2}]}(u)]$  for all  $u \leq \frac{1}{2}$  and  $h(d_2^0 x, u, t) = [\zeta_{d_2^0 x, t}([\frac{1}{2}, 1]), h_{d_2^0 x, t, [\frac{1}{2}, 1]}(u)]$  for all  $u \geq \frac{1}{2}$ .

Let  $y \in Y$ . For all  $t \in I$  we set  $\mathcal{A}_{y, t} = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$  and  $\zeta_{y, t}([0, \frac{1}{2}]) = y$ . For  $t \leq \frac{1}{3}$  we set  $\zeta_{y, t}([\frac{1}{2}, 1]) = y$ ,  $h_{y, t, [0, \frac{1}{2}]}(u) = (1+3t)u$ , and  $h_{y, t, [\frac{1}{2}, 1]}(u) = (1-3t)u + 3t$ . For  $t > \frac{1}{3}$  we set  $\zeta_{y, t}([\frac{1}{2}, 1]) = d_1^1 x$  and  $h_{y, t, [0, \frac{1}{2}]}(u) = 2u$ . For  $\frac{1}{3} < t \leq \frac{2}{3}$  we set  $h_{y, t, [\frac{1}{2}, 1]}(u) = (3t-1)(2u-1)$  and for  $t > \frac{2}{3}$  we set  $h_{y, t, [\frac{1}{2}, 1]}(u) = 2u-1$ . Then all maps  $h_{y, t, [0, \frac{1}{2}]}$  and  $h_{y, t, [\frac{1}{2}, 1]}$  are order-preserving and for each  $t \in I$ ,  $h(y, u, t) = [\zeta_{y, t}([0, \frac{1}{2}]), h_{y, t, [0, \frac{1}{2}]}(u)]$  ( $u \leq \frac{1}{2}$ ) and  $h(y, u, t) = [\zeta_{y, t}([\frac{1}{2}, 1]), h_{y, t, [\frac{1}{2}, 1]}(u)]$  ( $u \geq \frac{1}{2}$ ).

Let  $z \in R_r$  ( $r \geq 1$ ) and  $t \in I$ . We set  $\mathcal{A}_{z, t} = \{I^r\}$ ,  $\zeta_{z, t}(I^r) = z$ , and  $h_{z, t, I^r} = id_{I^r}$ . Then  $h_{z, t, I^r}$  is order-preserving and satisfies  $h(z, (u_1, \dots, u_r), t) = [\zeta_{z, t}(I^r), h_{z, t, I^r}(u_1, \dots, u_r)]$ . This ends the verification of condition 4.4(ii).

By 4.4, a dihomotopy  $H: |P| \times I \rightarrow |P|$  is given by  $H([z, (u_1, \dots, u_r)], t) = h(z, (u_1, \dots, u_r), t)$ . By definition,  $H([z, (u_1, \dots, u_r)], t) = [z, (u_1, \dots, u_r)]$  for all  $r \geq 0$ ,  $z \in R_r$ ,  $(u_1, \dots, u_r) \in I^r$ , and  $t \in I$ . One easily checks that  $H([z, (u_1, \dots, u_r)], 0) = [z, (u_1, \dots, u_r)]$  for all  $r \geq 0$ ,  $z \in P_r$ , and  $(u_1, \dots, u_r) \in I^r$ . We have

$$h(x, (u_1, u_2), 1) = \begin{cases} [d_1^0 x, u_1 + u_2], & u_1 \leq u_2, u_2 \leq 1 - u_1, \\ [d_2^1 x, u_1 + u_2 - 1], & u_1 \leq u_2, u_2 \geq 1 - u_1, \\ [d_1^0 x, 2u_1], & u_1 \geq u_2, u_1 \leq \frac{1}{2}, \\ [d_2^1 x, 2u_1 - 1], & u_1 \geq u_2, u_1 \geq \frac{1}{2} \end{cases}$$

and

$$h(y, u, 1) = \begin{cases} [y, 2u], & u \leq \frac{1}{2}, \\ [d_1^1 x, 2u - 1], & u \geq \frac{1}{2} \end{cases}$$

and thus  $H([z, (u_1, \dots, u_r)], 1) \in |Q|$  for all  $r \geq 0$ ,  $z \in P_r$ , and  $(u_1, \dots, u_r) \in I^r$ . We have

$$h(x, (1, u), t) = \begin{cases} [d_1^1 x, u], & t \leq \frac{1}{3}, \\ [d_1^1 x, 3t - 1 + (2 - 3t)u], & \frac{1}{3} \leq t \leq \frac{2}{3}, \\ [d_1^1 x, 1], & t \geq \frac{2}{3}. \end{cases}$$

It follows that  $H([z, (u_1, \dots, u_r)], t) \in |Q|$  for all  $r \geq 0$ ,  $z \in Q_r$ ,  $(u_1, \dots, u_r) \in I^r$ , and  $t \in I$ . This implies that the inclusion  $\iota: |Q| \hookrightarrow |P|$  is a dihomotopy equivalence relative to  $|R|$ . Indeed, by 2.10 and 2.11, the map  $f: |P| \rightarrow |Q|$ ,  $f([z, (u_1, \dots, u_r)]) = H([z, (u_1, \dots, u_r)], 1)$  is a d-map. A dihomotopy relative to  $|R|$  from  $id_{|Q|}$  to  $f \circ \iota$  is given by  $G: |Q| \times I \rightarrow |Q|$ ,  $G([z, (u_1, \dots, u_r)], t) = H([z, (u_1, \dots, u_r)], t)$  and  $H$  is a dihomotopy relative to  $|R|$  from  $id_{|P|}$  to  $\iota \circ f$ .

Suppose now that  $Y \neq \emptyset$ . Since  $Q_0 = P_0$  and  $Q_1 \subseteq P_1$ , we have  $Extrl(P) \subseteq Extrl(Q)$ . Let  $v \in Q_0 = P_0$  be a minimal element of  $Q$ . Then  $d_1^1 z \neq v$  for all  $z \in Q_1 = P_1 \setminus \{d_2^0 x\}$ . Since  $Y \neq \emptyset$ , there exists an element  $y \in P_1 \setminus \{d_2^0 x\}$  such that  $d_1^1 y = d_1^1 d_2^0 x$ . Therefore  $d_1^1 d_2^0 x \neq v$  and  $v$  is minimal in  $P$ . Let  $v \in Q_0 = P_0$  be a maximal element of  $Q$ . Then  $d_1^0 z \neq v$  for all  $z \in Q_1 = P_1 \setminus \{d_2^0 x\}$ . We have  $d_1^0 d_2^0 x = d_1^0 d_1^1 x$ .

Since  $x$  is regular,  $d_1^0 x \neq d_2^0 x$ . Therefore  $d_1^0 d_2^0 x \neq v$  and  $v$  is maximal in  $P$ . It follows that  $\text{Extrl}(Q) \subseteq \text{Extrl}(P)$  and hence that  $\text{Extrl}(Q) = \text{Extrl}(P)$ . Clearly,  $d_1^1 d_2^0 x = d_1^0 d_1^1 x \notin \text{Extrl}(P)$  and therefore  $\text{Extrl}(P) \subseteq R$ . It follows now from 3.7 that  $\iota$  induces an isomorphism of fundamental bipartite graphs  $\tilde{\pi}_1(|Q|, \text{Extrl}(|Q|)) \xrightarrow{\cong} \tilde{\pi}_1(|P|, \text{Extrl}(|P|))$ . This terminates the proof of the theorem in the case  $a = 1$  and  $b = 0$ .

Consider now the case  $a = 1$  and  $b = 1$ . Then  $P^{op}$  satisfies the conditions of the theorem in the case  $a = 1$  and  $b = 0$ . It follows that the graded sets  $Q$  and  $R$  define precubical subsets of  $P^{op}$  and hence of  $P$ . We denote the precubical subsets of  $P$  defined by the graded sets  $Q$  and  $R$  by  $Q$  and  $R$ . Then the precubical subsets of  $P^{op}$  defined by the graded sets  $Q$  and  $R$  are  $Q^{op}$  and  $R^{op}$ . We note that  $R^{op}$  is a precubical subset of  $Q^{op}$  and that  $R$  is a precubical subset of  $Q$ . By the theorem in the case  $a = 1$  and  $b = 0$ , the inclusion  $|Q^{op}| \hookrightarrow |P^{op}|$  is a dihomotopy equivalence relative to  $|R^{op}|$ . It follows from 3.8(i) that the inclusion  $\iota: |Q| \hookrightarrow |P|$  is a dihomotopy equivalence relative to  $|R|$ . If  $Y \neq \emptyset$ , then by the theorem in the case  $a = 1$  and  $b = 0$ ,  $\text{Extrl}(P^{op}) = \text{Extrl}(Q^{op}) \subseteq R^{op}$ . It follows that  $\text{Extrl}(P) = \text{Extrl}(Q) \subseteq R$  and hence, by 3.7, that  $\iota$  induces an isomorphism of fundamental bipartite graphs  $\tilde{\pi}_1(|Q|, \text{Extrl}(|Q|)) \xrightarrow{\cong} \tilde{\pi}_1(|P|, \text{Extrl}(|P|))$ . This terminates the proof of the theorem in the case  $a = 1$  and  $b = 1$  and hence in the case  $a = 1$ .

The theorem in the remaining case  $a = 2$  is deduced from the theorem in the case  $a = 1$  using transposed precubical sets in the same way as the theorem in the case  $a = 1$  and  $b = 1$  is deduced from the theorem in the case  $a = 1$  and  $b = 0$  using opposite precubical sets.  $\square$

**Theorem 6.2.** *Let  $P$  be a precubical set,  $b \in \{0, 1\}$ , and  $x \in P_2$  be a regular element such that no element of  $P_2 \setminus \{x\}$  has  $d_1^{1-b}x$  or  $d_2^b x$  in its boundary and the only elements of  $P_1$  having  $d_1^{1-b}d_2^b x = d_1^b d_1^{1-b}x$  in their boundary are  $d_1^{1-b}x$  and  $d_2^b x$ . Then a precubical subset  $Q$  of  $P$  such that the inclusion  $\iota: |Q| \hookrightarrow |P|$  is a dihomotopy equivalence relative to  $|Q|$  is given by  $Q_0 = P_0 \setminus \{d_1^{1-b}d_2^b x\}$ ,  $Q_1 = P_1 \setminus \{d_1^{1-b}x, d_2^b x\}$ ,  $Q_2 = P_2 \setminus \{x\}$ , and  $Q_r = P_r$  ( $r > 2$ ). Moreover,  $\text{Extrl}(P) = \text{Extrl}(Q)$  and  $\iota$  induces an isomorphism of fundamental bipartite graphs  $\tilde{\pi}_1(|Q|, \text{Extrl}(|Q|)) \xrightarrow{\cong} \tilde{\pi}_1(|P|, \text{Extrl}(|P|))$ .*

*Proof.* The precubical set  $P$ , the number  $b$ , and the element  $x$  satisfy the conditions of 6.1 with  $a = 1$ . It follows that a precubical subset  $M$  of  $P$  such that  $Q$  is a precubical subset of  $M$  and the inclusion  $|M| \hookrightarrow |P|$  is a dihomotopy equivalence relative to  $|Q|$  is given by  $M_0 = P_0$ ,  $M_1 = P_1 \setminus \{d_2^b x\}$ ,  $M_2 = P_2 \setminus \{x\}$ , and  $M_r = P_r$  ( $r > 2$ ). The precubical set  $M$ , the number  $1 - b$ , and the element  $d_1^{1-b}x$  satisfy the conditions of 5.1. We have  $Y = \{y \in M_1 \mid d_1^{1-b}y = d_1^b d_1^{1-b}x\} = \emptyset$ . It follows that the inclusion  $|Q| \hookrightarrow |M|$  is a dihomotopy equivalence relative to  $|Q|$  and hence that the inclusion  $\iota: |Q| \hookrightarrow |P|$  is a dihomotopy equivalence relative to  $|Q|$ . Note that  $d_1^{1-b}d_2^b x = d_1^b d_1^{1-b}x \notin \text{Extrl}(P)$ . Therefore  $\text{Extrl}(P) \subseteq Q$ . This implies that  $\text{Extrl}(P) \subseteq \text{Extrl}(Q)$ . Let  $v \in Q_0$  be a minimal element of  $Q$ . Then for all  $z \in Q_1$ ,  $d_1^1 z \neq v$ . If  $b = 0$ , then  $d_2^1 x \in Q_1$  and therefore  $d_1^1 d_1^{1-b}x = d_1^1 d_2^1 x \neq v$ . If  $b = 1$ , then  $d_1^1 d_1^{1-b}x = d_1^{1-b}d_2^1 x \notin Q_0$  and therefore  $d_1^1 d_1^{1-b}x \neq v$ . If  $b = 0$ , then  $d_1^1 d_2^b x = d_1^{1-b}d_2^b x \notin Q_0$  and therefore  $d_1^1 d_2^b x \neq v$ . If  $b = 1$ , then  $d_1^1 x \in Q_1$  and

therefore  $d_1^1 d_2^b x = d_1^1 d_1^1 x \neq v$ . It follows that  $v$  is a minimal element of  $P$ . A similar argument shows that any maximal element of  $Q$  is also a maximal element of  $P$ . It follows that  $\text{Extrl}(P) = \text{Extrl}(Q)$  and hence, by 3.7, that  $\iota$  induces an isomorphism of fundamental bipartite graphs  $\pi_1(|Q|, \text{Extrl}(|Q|)) \xrightarrow{\cong} \pi_1(|P|, \text{Extrl}(|P|))$ .  $\square$

## 7. EXAMPLES

In this section, we use our reduction techniques to compute small models for three simple precubical sets. In each case, we know *a priori* that the geometric realizations of the model and the given precubical set are dihomotopy equivalent relative to the extremal elements and have isomorphic fundamental bipartite graphs.

**Example 7.1.** Consider the 2-dimensional precubical set depicted in figure 2(a) below. The grey squares represent the elements of degree 2, the arrows represent the elements of degree 1, and the end points of the arrows represent the elements of degree 0. The arrow corresponding to an edge  $x$  points from  $d_1^0 x$  to  $d_1^1 x$ . The left-hand edge of a square  $x$  is  $d_1^0 x$ , the right-hand edge is  $d_1^1 x$ , the lower edge is  $d_2^0 x$ , and the upper edge is  $d_2^1 x$ . We use the following sequence of 2-dimensional reductions to deform this precubical set into the 1-dimensional precubical subset depicted in figure 2(b): We proceed linewise from the top left square to the bottom right square using Theorem 6.2 with  $b = 1$  to eliminate all squares except for the four squares on the right of the holes where we use Theorem 6.1 with  $a = 2$  and  $b = 0$  and the four squares below the holes where we use Theorem 6.1 with  $a = 1$  and  $b = 1$ . A sequence of 1-dimensional reductions using Theorem 5.1 with  $b = 0$  permits us to simplify the model further to the 1-dimensional precubical set with four vertices in figure 2(c). From this we obtain the final model in figure 2(d) using Theorem 5.1 twice with  $b = 1$ .

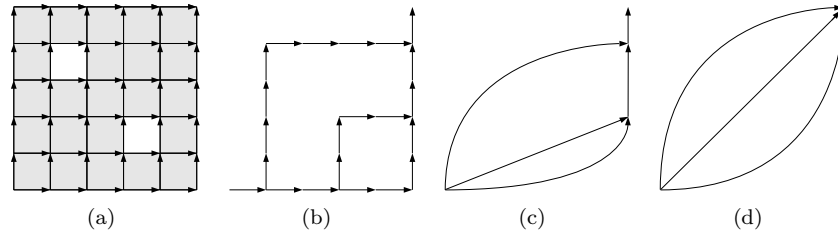


FIGURE 2.

**Example 7.2.** Consider the precubical set in figure 3(a). We use the following sequence of 2-dimensional reductions to deform this precubical set into the 1-dimensional precubical subset in figure 3(b): We proceed linewise from the top left square to the square on the left of the lower hole using Theorem 6.2 with  $b = 1$  to eliminate all squares except for the one on the right of the upper hole where we use Theorem 6.1 with  $a = 2$  and  $b = 0$  and the one below the upper hole where we use Theorem 6.1 with  $a = 1$  and  $b = 1$ . We then eliminate the remaining squares using Theorem 6.2 with  $b = 0$  proceeding linewise upwards from the bottom right square to the square to the right of the lower hole. We simplify the model further to the precubical set in figure 3(c) by means of a sequence of 1-dimensional reductions

using Theorem 5.1 with  $b = 0$ . Using Theorem 5.1 with  $b = 1$  we finally obtain the model in figure 3(d).

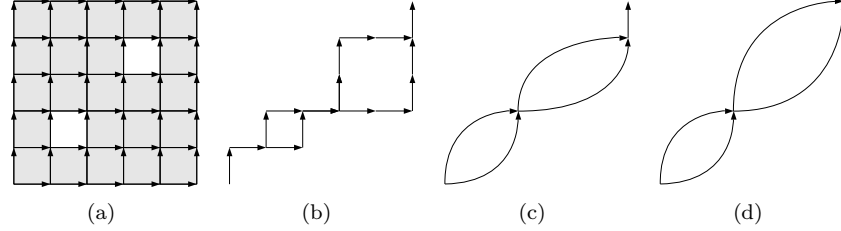


FIGURE 3.

**Example 7.3.** We deform the *Swiss flag* in figure 4(a) into the 1-dimensional precubical subset in figure 4(b) successively as follows: We proceed linewise from the top left square to the bottom right square using Theorem 6.2 with  $b = 1$  to eliminate all squares except for the three squares on the right of the upper and middle holes where we use Theorem 6.1 with  $a = 2$  and  $b = 0$  and the three squares below the left and the middle holes where we use Theorem 6.1 with  $a = 1$  and  $b = 1$ . Using Theorem 5.1 several times with  $b = 0$  we obtain the 1-dimensional precubical set in figure 4(c). We finally obtain the model in figure 4(d) using Theorem 5.1 twice with  $b = 1$ .

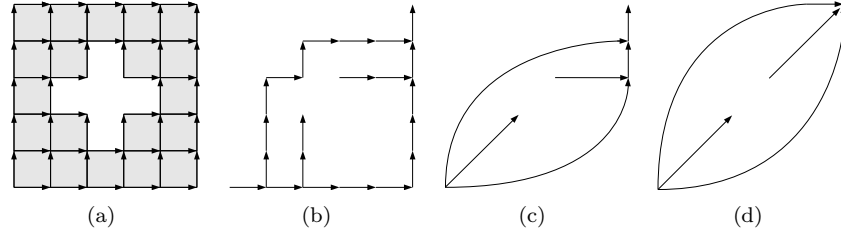


FIGURE 4.

**Remark 7.4.** In [1], P. Bubenik introduces *extremal models* of d-spaces and calculates such extremal models for the geometric realizations of the precubical set of the introduction and the precubical sets of Examples 7.2 and 7.3. In all cases, the extremal model is a full subcategory of the fundamental category of the geometric realization of our small model, namely the full subcategory generated by the vertices of the model. It would be interesting to know whether this link between the models constructed using our approach and the extremal models of [1] can be established in general.

## REFERENCES

- [1] P. Bubenik, Models and Van Kampen theorems for directed homotopy theory, *Homology, Homotopy and Applications* 11(1) (2009), 185-202.
- [2] L. Fajstrup, Dipaths and dihomotopies in a cubical complex, *Advances in Applied Mathematics* 35 (2005), 188-206.

- [3] L. Fajstrup, M. Raußen, E. Goubault, Algebraic topology and concurrency, *Theoretical Computer Science* 357 (2006), 241-278.
- [4] L. Fajstrup, M. Raußen, E. Goubault, E. Haucourt, Components of the Fundamental Category, *Applied Categorical Structures* 12 (2004), 81-108.
- [5] P. Gaucher and E. Goubault, Topological deformation of higher dimensional automata, *Homology, Homotopy and Applications* 5 (2) (2003), 39-82.
- [6] E. Goubault, Some geometric perspectives in concurrency theory, *Homology, Homotopy and Applications* 5 (2) (2003), 95-136.
- [7] E. Goubault, Cubical Sets are Generalized Transition Systems, preprint (2002), available at <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.23.7908>
- [8] E. Goubault, E. Haucourt, Components of the Fundamental Category II, *Applied Categorical Structures* 15 (2007), 387-414.
- [9] M. Grandis, *Directed Algebraic Topology - Models of Non-Reversible Worlds*, New Mathematical Monographs 13, Cambridge University Press (2009).
- [10] M. Raußen, Invariants of Directed Spaces, *Applied Categorical Structures* 15 (2007), 355-386.
- [11] G. Winskel, M. Nielsen, Models for concurrency, *Handbook of logic in computer science (vol. 4): semantic modelling*, Oxford University Press (1995), 1-148.

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